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Lecture Notes QED 2012

"QED Theory"

Cargèse

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Lit.: W. Dittrich, H. Gies, "Probing the quantum vacuum", Springer Tracts in Modern Physics 166 (2000)

1. Introduction

QED : Quantum field theory (QFT)
(merger of quantum mechanics & special relativity)

degrees of freedom:

- fermionic electron / positron field $\Psi(x)$:

$\bar{\Psi}(x) = \Psi^{+}\gamma^{0}$ (Dirac conjugate)

Dirac (4-component complex) spinor field

- photon field $A_\mu(x)$: real $U(1)$ gauge boson

microscopic Lagrangian

$$\mathcal{L}_{QED} = \bar{\Psi}\, i\gamma_\mu (\partial^\mu - ieA^\mu)\Psi - m\bar{\Psi}\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1}$$

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conventions: $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$, $g = \mathrm{diag}(-1,1,1,1)$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{2}$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] , \quad \text{e.g. } \sigma^{12} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

field strength:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{3}$$

gauge symmetry:

$$\begin{aligned} \Psi(x) &\to e^{ie\lambda(x)} \Psi(x) \\ \bar{\Psi}(x) &\to \bar{\Psi}(x)\, e^{-ie\lambda(x)} \\ A^\mu(x) &\to A^\mu(x) + \partial^\mu \lambda(x) \end{aligned} \tag{4}$$

covariant derivative

$$D^\mu = \partial^\mu - ie A^\mu$$

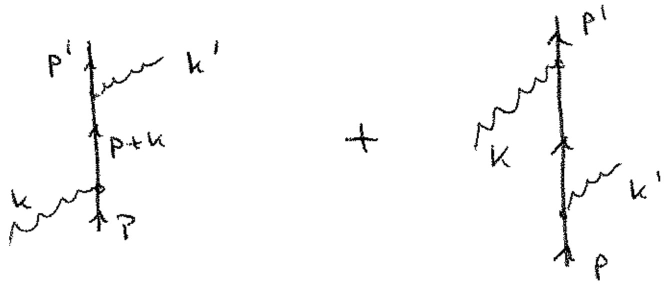
(5)

Dirac operator : $\not{D} = \gamma_\mu D^\mu$

Brief summary of "textbook" QED phenomena
(not to be discussed in detail in this course)

(1) Compton scattering, Klein-Nishina formula

$$e^- \gamma \rightarrow e^- \gamma$$



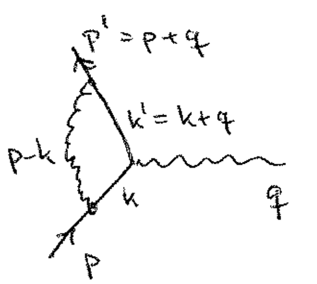
limit $\omega \rightarrow 0$ $\quad (p\cdot k = m\omega)$: Thomson cross section

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(1+\cos^2\theta\right), \quad \sigma_{tot} = \frac{8\pi\alpha^2}{3m^2} \qquad (6)$$

generalizations : non-linear Compton scattering
(cf. talk by A. Ilderton)

(2) Anomalous magnetic moment of the electron
(electron g-2) $\qquad \vec{\mu} = g\left(\frac{e}{2m}\right)\vec{S}$

(7)



$$\Longrightarrow \quad g = \underset{\substack{\uparrow \\ \text{Dirac}}}{2} + \underset{\substack{\uparrow \\ \text{Schwinger}}}{\frac{\alpha}{\pi}} + \mathcal{O}\left(\left(\frac{\alpha}{\pi}\right)^2\right)$$

One of the most stringent tests of QED:

experimentally determined to ~ 8 parts in 10^{13}

$\Rightarrow$ determines α to within 0.7 ppb

(in agreement with all other exp.)

(3) Bound-state QED (cf. C. Pachucki's lectures)

$\Rightarrow$ "Best Tested Theory"

Why further tests?

So far: – low amplitude tests (one, two, ... photon couplings)

– high momentum tests

missing: high amplitude / field tests

 (potentially non-perturbative in α)

2. Strong-field QED & Heisenberg-Euler action

some notation: classical electrodynamics (in vacuum)

Lagrangian (Maxwell)

$$\mathcal{L}_M = -\frac{1}{2}\left(\vec{B}^2 - \vec{E}^2\right) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Eq. of motion

$$0 = \underbrace{\frac{\partial \mathcal{L}_M}{\partial A_\nu}}_{=0} - \partial_\mu \underbrace{\frac{\partial \mathcal{L}_M}{\partial(\partial_\mu A_\nu)}}_{=2\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}} \tag{8}$$

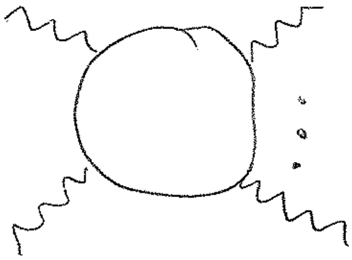
$$\Rightarrow \qquad \partial_\mu F^{\mu\nu} = 0 \qquad \text{Maxwell's equations}$$

- determines electromagnetic (EM) fields
- linear equations $\Rightarrow$ superposition principle

BUT: QED

Heisenberg's uncertainty principle allows for fluctuations

charged fluctuations induce interactions between EM fields



$\Rightarrow$ effective description in terms of an effective Lagrangian (Heisenberg–Euler 1936)

$\mathcal{L}_{HE} \rightarrow \mathcal{L}_M$ in the classical limit ($\hbar \to 0$)

requirements $\mathcal{L}_{HE}$: – Lorentz invariant, Lorentz scalar

– gauge invariant

– CP invariant

– mass dimension $0 = [S] = [\int d^4x \, \mathcal{L}]$

$\Rightarrow \quad [\mathcal{L}] = 4 \quad (\hbar = c = 1)$

use field strength invariants

$$\mathcal{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}(\vec{B}^2 - \vec{E}^2)$$

$$\mathcal{G} = \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} = -\vec{E}\cdot\vec{B} \quad (\text{CP odd}) \tag{9}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$.

Expectation:

$$\mathcal{L}_{HE} = \mathcal{L}_{HE}(\mathcal{F}, \mathcal{G}^2) + \text{derivative terms}$$

e.g.

$$= -\mathcal{F} + c_1 \mathcal{F}^2 + c_2 \mathcal{G}^2 + d_1 F_{\mu\nu} \partial^2 F^{\mu\nu}$$

$$+ \mathcal{O}(\mathcal{F}^3, \mathcal{G}^2\mathcal{F}, \text{further derivative terms})$$

mass dimension $[c_{1,2}] = -4$, $[d_1] = -2$

Which dimensionful parameter sets the scale for $c_{1,2}, d_1$?

$\Rightarrow$ Scale of fluctuations:

m: electron mass

$$m \simeq 511 \text{ keV}$$

$$\hat{=} \; 7.6 \cdot 10^{11} \text{ GHz} \; \hat{=} \; (3.8 \cdot 10^{-13} \text{m})^{-1} = \frac{1}{\lambda_c}$$

Compton wavelength

$$\frac{m^2}{e} \; \hat{=} \; 4 \cdot 10^9 \text{ Tesla} \qquad (10)$$

If $F_{\mu\nu}$ is slowly varying in comparison with the Compton time / length λ_c, $F_{\mu\nu} \simeq$ const. is a very good approximation

(typically very well satisfied by laboratory magnets, cavities, optical lasers ...)

$$\Rightarrow \mathcal{L}_{HE} \simeq \mathcal{L}_{HE}(\mathcal{F}, \mathcal{G}^2)$$

If $|\vec{E}|, |\vec{B}| \ll E_{cr} = \frac{m^2}{e}$, $\mathcal{L}_{HE} \simeq \mathcal{L}_M = -\mathcal{F}$

(classical limit)

critical field strength: $m c^2 \overset{!}{=} e E_{cr} \lambda_c$ (11)

Sketch of the calculation

Use toy model : "scalar QED" (neglect electron spin)

QFT : ($\mathcal{L}^1_{HE}$: 1-loop contribution to $\mathcal{L}_{HE}$)

$$e^{i\int d^4x\, \mathcal{L}^1_{HE}} = \underset{\text{normalization}}{\mathcal{N}} \underset{\substack{\text{integral over all}\\ \text{field config's}}}{\int \mathcal{D}\phi}\; e^{i\int d^4x \left[\phi^*(-D^2[A])\phi + m^2\phi^*\phi\right]}$$

ϕ : scalar electron field $\in \mathbb{C}$ (charged Klein-Gordon field)

$$= \mathcal{N} \det{}^{-1}\left(-D^2[A] + m^2\right) \qquad (12)$$

(where the functional integration is analogous to the complex Gaußian integral $\int dz\, d\bar{z}\; e^{-\bar{z} a z} \sim \frac{1}{a}$)

$$\Rightarrow \int d^4x\, \mathcal{L}^1_{HE} = i \ln \det\left(-D^2 + m^2\right) - \text{normaliz.}$$

$$= i \operatorname{Tr} \ln\left(-D^2 + m^2\right) - \text{normaliz.}$$

$$\underset{\text{Frullani's formula}}{=} -i \int_0^\infty \frac{dT}{T} \operatorname{Tr} e^{-T(-D^2+m^2)} - \text{norm.} \qquad (13)$$

Simplify to const. B-field in z direction:

$$A_\mu = \frac{1}{2} B \begin{pmatrix} 0 \\ -y \\ x \\ 0 \end{pmatrix} \qquad (14)$$

$\Rightarrow$ eigenvalues of $-D^2$: $\quad -p_0^2 + p_z^2 + \underbrace{eB(2n+1)}_{\text{Landau-levels}}$

(cf. quantum mechanical particle in B-field)

p_0^2: energy $\qquad p_z^2$: z-momentum

$$\int d^4x\, \mathcal{L}^1_{HE} = -i \int_0^\infty \frac{dT}{T} \; e^{-m^2 T} \; \mathrm{Tr}\, e^{-(-D^2)T}$$

Wick rotation

$p_0 \Rightarrow i p_E$

$$= -i \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{-\infty}^{\infty} \frac{i\, dp_E}{2\pi/L} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi/L} \sum_{n=0}^{\infty} g(n)\, e^{-p_E^2 T}\, e^{-p_z^2 T}\, e^{-eB(2n+1)T} \qquad (15)$$

$g(n)$: density of Landau level states

must be such that

$$\sum_{n=0}^{\infty} g(n)\, f(eB(2n+1)) \xrightarrow{B\to 0} \int \frac{dp_x dp_y}{(2\pi/L)^2} f(p_x^2 + p_y^2) \qquad (16)$$

Riemann sum $\Rightarrow$ $\quad g(n) = L^2 \dfrac{eB}{2\pi}$

$$\overset{(15)}{\Rightarrow} \quad \int d^4x\, \mathcal{L}^1_{HE} = \Omega \int_0^\infty \frac{dT}{T} e^{-m^2 T} \frac{1}{(4\pi T)} \frac{eB}{2\pi} \underbrace{\sum_{n=0}^{\infty} \left(e^{-2eBT}\right)^n e^{-eBT}}_{= \frac{e^{eBT}}{2\sinh eBT}}$$

$$\Omega = L^4 = \int d^4x$$

$$\Vert$$

$$\Omega \mathcal{L}^1_{HE} = \frac{\Omega}{16\pi^2} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} \underbrace{\frac{eBT}{\sinh eBT}}_{\text{contains all powers in } (eB)^2} \tag{17}$$

Result:

$$\underset{\text{scalar QED}}{\mathcal{L}^1_{HE}} = +\frac{1}{16\pi^2} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} \frac{eBT}{\sinh eBT} \tag{18}$$

Spinor QED:

$$-D^2 \to \cancel{D}^2 = -D^2 - i\sigma^{\mu\nu} D_\mu D_\nu = -D^2 - \frac{e}{2} \sigma_{\mu\nu} F^{\mu\nu} \tag{19}$$

$$\Rightarrow e^{-\cancel{D}^2 T} \xrightarrow{B \text{ const.}} e^{-D^2 T} e^{e\sigma_3 BT} \tag{20}$$

Pauli term

Spinor QED Heisenberg - Euler

$$\mathcal{L}^{1}_{HE} = \underset{\substack{\uparrow \\ \text{Fermi} \\ \text{statistics}}}{-}\ \underset{\substack{\uparrow \\ \text{Spin} \\ \text{degrees}}}{2}\ \frac{1}{16\pi^2}\int_0^\infty \frac{dT}{T^3}\, e^{-m^2T}\, \frac{eBT}{\sinh eBT}\ \underset{\substack{\uparrow \\ \text{from Pauli term}}}{\cosh eBT}$$

$$= -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\, e^{-m^2T}\, eBT \coth eBT \qquad (21)$$

(unrenormalized)

Full Heisenberg - Euler result for $\vec{E}, \vec{B} \neq 0$

$$\mathcal{L}^{1}_{HE} = -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\, e^{-m^2T}\Big\{ (eT)^2 |\mathcal{G}| \coth\Big[eT\Big(\sqrt{\mathcal{F}^2+\mathcal{G}^2}+\mathcal{F}\Big)^{1/2}\Big] \cdot \cot\Big[eT\Big(\sqrt{\mathcal{F}^2+\mathcal{G}^2}-\mathcal{F}\Big)^{1/2}\Big] - \frac{2}{3}(eT)^2\mathcal{F} - 1 \Big\} \qquad (22)$$

$-\frac{2}{3}(eT)^2\mathcal{F}$: charge renormalization counter-term

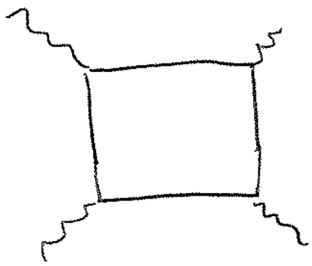
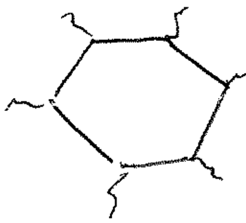
(non-trivial, requires renormalization theory)

-1: zero-field subtraction from normalization

(trivial)

Weak-field expansion:

$$\mathcal{L}^{1}_{HE} \simeq \underbrace{\frac{8}{45}\frac{\alpha^2}{m^4}}_{\equiv c_1}\mathcal{F}^2 + \underbrace{\frac{14}{45}\frac{\alpha^2}{m^4}}_{\equiv c_2}\mathcal{G}^2 + \mathcal{O}\left(\frac{F^6}{m^8}\right) \qquad (23)$$

$\hat{=}$  $+$  $+ \ldots$

- $\mathcal{L}_{HE}$ translates quantum corrections to Maxwell's theory into a classical language, giving rise to "classical" non-linearities

- c_1 & c_2 are suppressed by α^2

3 Phenomena (Quantum vacuum physics)

3.1 Light propagation in a strong EM field

Weak-field HE-action

$$\mathcal{L}_{HE} = -\mathcal{F} + c_1 \mathcal{F}^2 + c_2 \mathcal{G}^2 \quad , c_1 = \frac{8}{45}\frac{\alpha^2}{m^4} , c_2 = \frac{14}{45}\frac{\alpha^2}{m^4} \tag{24}$$

EoM $\Rightarrow$

$$0 = -2\, \partial_\mu \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} \quad , \frac{\partial \mathcal{F}}{\partial F_{\alpha\beta}} = \frac{1}{2} F^{\alpha\beta} , \frac{\partial \mathcal{G}}{\partial F_{\alpha\beta}} = \frac{1}{2} \widetilde{F}^{\alpha\beta}$$

$$= \partial_\mu \left(F^{\mu\nu} - 2c_1 \mathcal{F} F^{\mu\nu} - 2c_2 \mathcal{G} \widetilde{F}^{\mu\nu} \right) \tag{25}$$

Let us decompose the field strength into a strong background field $\overline{F}^{\mu\nu} \simeq$ const. and a weak probe field f : $F^{\mu\nu} = \overline{F}^{\mu\nu} + f^{\mu\nu}$

$$\Rightarrow \quad 0 = \partial_\mu f^{\mu\nu} - c_1 \overline{F}_{\alpha\beta} \overline{F}^{\mu\nu} \partial_\mu f^{\alpha\beta} - c_2 \widetilde{\overline{F}}_{\alpha\beta} \widetilde{\overline{F}}^{\mu\nu} \partial_\mu f^{\alpha\beta} + \mathcal{O}(f^2, c_{1,2}^2) \tag{26}$$

(drop bar from now on)

Assume Fourier decomposition

$$f^{\mu\nu} \sim k^\mu a^\nu - k^\nu a^\mu \tag{27}$$

& Lorentz gauge $\quad k_\mu a^\mu = 0$

$$\overset{(C.3)}{\Rightarrow} \quad 0 = k^2 a^\nu - 2c_1 F_{\alpha\beta} F^{\mu\nu} k_\mu k^\alpha a^\beta - 2c_2 \tilde{F}_{\alpha\beta} \tilde{F}^{\mu\nu} k_\mu k^\alpha a^\beta \tag{28}$$

check: in the classical limit $c_{1,2} \to 0$, we rediscover $k^2 a^\nu = 0$, i.e. $k^2 = \vec{k}^2 - \omega^2 = 0$ (29)
implying that light propagates "on the light cone".
with speed $v = \frac{\omega}{|\vec{k}|} = c = 1$ (phase velocity).

Now: (C.5) is a matrix equation with possibly different solutions.

Ansatz: $a_\parallel^\mu \sim \tilde{F}^{\mu\nu} k_\nu \equiv (\tilde{F}k)^\mu$, $a_\perp^\mu = (Fk)^\mu$

$$\Rightarrow a_\parallel^\mu : \quad 0 = \big(k^2 - 2c_2 \underbrace{(\tilde{F}k)^2}_{=(Fk)^2} + 2c_2 \mathfrak{F} k^2\big)(\tilde{F}k)^\nu - (2c_1 \mathcal{G} k^2)\, Fk^\nu$$

$$a_\perp^\mu : \quad 0 = \big(k^2 - 2c_1 (Fk)^2\big) Fk^\nu - (2c_2 \mathcal{G} k^2)(\tilde{F}k)^\nu \tag{30}$$

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using
$$F^{\mu\alpha} F^\nu{}_\alpha - \tilde{F}^{\mu\alpha} \tilde{F}^\nu{}_\alpha = 2\mathfrak{F} g^{\mu\nu}$$
$$F^{\mu\alpha} \tilde{F}^\nu{}_\alpha = \tilde{F}^{\mu\alpha} F^\nu{}_\alpha = \mathcal{G} g^{\mu\nu} \tag{31}$$

We observe that

$$k^2 = 0 + \mathcal{O}(c_{1,2})$$

is a consistent solution

To order $\mathcal{O}(c_{1,2})$ the dispersion relations (light cone conditions) read

$$a_\parallel^\mu : k^2 = 2c_2 (Fk)^2 \quad , \quad a_\perp^\mu : k^2 = 2c_1 (Fk)^2 \qquad (32)$$

More explicitly, let $F^{\mu\nu}$ describe a const. magnetic field, then

$$(Fk)^2 = (F^{\mu\nu} k_\nu)^2 \triangleq |\vec{k}|^2 B^2 \sin^2 \underbrace{\theta_B}_{\sphericalangle(\vec{B},\vec{k})} \qquad (33)$$

$\Rightarrow$ $a_\perp^\mu \sim Fk^\mu$ is polarized $\perp$ WRT $(\vec{B},\vec{k})$ plane

$a_\parallel^\mu \sim \tilde{F}k^\mu$ " in the $(\vec{B},\vec{k})$ plane

Define phase velocity: $v = \frac{\omega}{|\vec{k}|}$, $k^\mu = (\omega, \vec{k})$

$$\Rightarrow \quad v_\parallel = 1 - c_2 \frac{(Fk)^2}{|\vec{k}|^2} = 1 - \frac{14}{45} \frac{\alpha^2}{m^4} B^2 \sin^2\theta_B$$

$$v_\perp = 1 - c_1 \frac{(\tilde{F}k)^2}{|\vec{k}|^2} = 1 - \frac{8}{45} \frac{\alpha^2}{m^4} B^2 \sin^2\theta_B \qquad (34)$$

$\Rightarrow$ - modified Quantum Vacuum is birefringent $n_{\parallel,\perp} = \frac{1}{v_{\parallel,\perp}}$

- Formulas valid in soft-photon limit $\omega \ll m$
& weak-field limit $B \ll \frac{m^2}{e}$

- dispersive effects are neglected $\mathcal{L}_{HE} = \mathcal{L}(\mathcal{F}, \mathcal{G}, \cancel{\partial \mathcal{F}})$
group velocity $\frac{d\omega}{d|\vec{k}|} = \frac{\omega}{|\vec{k}|}$ phase velocity for $\omega \ll m$

- $v_{\parallel,\perp} \leq 1$ since $c_1, c_2 > 0$
micro causality $\overset{?}{\Leftrightarrow}$ macrocausality

- experiment:
birefringence induces ellipticity

$$\Phi = 2\pi \, (n_\parallel - n_\perp) \, \frac{L}{\lambda} \tag{3.5.}$$

L ← optical path length
λ ← wave length

$$\Phi_{QED} = \frac{1}{15} \alpha \left(\frac{eB}{m^2}\right)^2 \frac{L}{\lambda} \left(1 + \underbrace{\frac{25}{4}\frac{\alpha}{\pi}}_{\text{2-loop} \simeq 1\%}\right)$$

strategies: use optical probe (λ fixed), optimize $B^2 L$ (PVLAS, BMV)
use high-intensity laser (L fixed), optimize $\frac{B^2}{\lambda}$ (Jena project)
(or: interferometry (GWI), or pulsar signal modulation (cf. A. Dupays))

- polarization sum rule:

$$v = 1 - \frac{1}{2}(c_1 + c_2)\frac{(Fk)^2}{|\vec{k}|^2} = 1 - \frac{11}{45}\frac{\alpha^2}{m^4} B^2 \sin^2\theta_B \qquad (36)$$

average over propagation direction:

$$v = 1 - \frac{44}{135}\frac{\alpha^2}{m^4}\frac{1}{2}B^2$$

$$= 1 - \frac{44}{135}\frac{\alpha^2}{m^4} u \qquad (37)$$

u: energy density of modified vacuum

Latorre, Pascual, Tarrach '95 "unified formula":

$$v = 1 - \frac{44}{135}\frac{\alpha^2}{m^4} u \qquad (38)$$

holds also for light propagation in

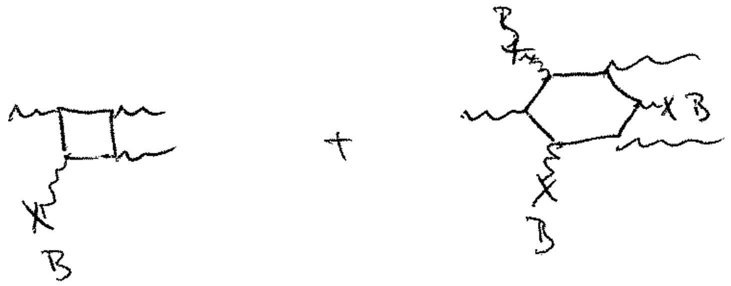
- thermal background $\quad u = \frac{\pi^2}{15} T^4 \quad$ (Barton)
- Casimir background $\quad u = -\frac{\pi^2}{720}\frac{1}{a^4} \quad$ (Scharnhorst)
- curved space (c.f. recent work by Shore, Hollowood)

3.2. Photon splitting (Toll '52, Adler '71)

process forbidden in vacuum due to Furry's theorem

$\sim$ tr odd # γ's $= 0$

in a B-field:



However: Box-Diagram contribution $= 0$ for $B = $ const.
(or order α supressed..)
(Adler's theorem due to kinematics)

Splitting matrix element, schematically

$$\mathcal{M} \simeq f_1 f_2 f_3 \left. \frac{\partial^3 \mathcal{L}_{HE}}{\partial F\, \partial F\, \partial F} \right|_{\bar{F}} \qquad (39)$$

particularity: polarization algebra allows only splitting process $(\perp \to \parallel + \parallel)$ ($\perp$ mode is "faster")

$\Rightarrow$ photon splitting could be an effective mechanism to produce polarized soft photons

absorption coefficient

$$\kappa = \frac{1}{32\pi\omega^2} \int_0^{\omega} d\omega_1 \int_0^{\omega} d\omega_2 \, \delta(\omega - \omega_1 - \omega_2) \, |\mathcal{M}|^2 \tag{40}$$

Heisenberg-Euler:

$$\kappa = \frac{\pi^2 \alpha^3}{15} \left(\frac{eB}{m^2}\right)^6 \left(\frac{\omega}{m}\right)^5 \sin^6\theta_B \left(\frac{\partial^3 \mathcal{L}}{\partial\mathcal{F}\partial\mathcal{G}^2}\Big|_B \right)^2 \frac{m^{16}}{e^{12}} \, m \tag{41}$$

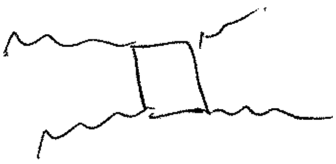
$$\begin{cases} \simeq 0.116 \text{ cm}^{-1} \left(\frac{eB}{m^2}\right)^6 \sin^6\theta_B \left(\frac{\omega}{m}\right)^5 , & \omega \ll m, \ B \ll \frac{m^2}{e} \\ \simeq 0.472 \text{ cm}^{-1} \sin^6\theta_B \left(\frac{\omega}{m}\right)^5 , & B \gg \frac{m^2}{e} \end{cases} \tag{42}$$

$1/\kappa$: absorption length

Consider Pulsar: $B \sim B_{cr} = \frac{m^2}{e}$

$\Rightarrow$ Photons with $\omega \lesssim m$ are polarized
(relevance for real observations unclear)

3.3 Light by light scattering (Euler '36, Karplus & Neuman '51)



HE - action:

$$\mathcal{L}_{HE} \simeq \frac{2}{45}\frac{\alpha^2}{m^4}\left(4\mathcal{F}^2 + 7\mathcal{G}^2\right) \qquad (43)$$

Matrix-element (depends on kinematics & polarizations)

$$\mathcal{M} \sim f_1 \dots f_4 \, \frac{\partial^4 \mathcal{L}}{\partial F \partial F \partial F \partial F} \qquad \text{(very schematically)}$$

differential cross section (pol. sum), $\frac{d\sigma}{d\Omega} = \sum \int_{\text{phase space}} |\mathcal{M}|^2$

$$\frac{d\sigma}{d\Omega} = \frac{139}{8100}\left(\frac{\alpha}{2\pi}\right)^2 r_0^{\,2} \left(\frac{\omega}{m}\right)^6 \left(3 + \cos^2\theta\right)^2 , \quad \omega \ll m \qquad (44)$$

where $r_0 = \frac{\alpha}{m}$ "classical electron radius"

Total cross section

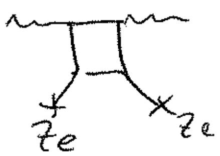
$$\sigma = \frac{973}{10125} \frac{\alpha^2}{\pi} r_0^2 \left(\frac{\omega}{m}\right)^6 \qquad (45.)$$

Cf. Thomson scattering

$$\frac{d\sigma}{d\Omega}(\omega) = \frac{1}{2} r_0^2 (1+\cos^2\theta) \quad , \sigma = \frac{8\pi}{3} r_0^2 \qquad (46.)$$

$\Rightarrow$ light-by-light is α^2 - suppressed & $\frac{\omega}{m}$ suppressed.

- direct measurement still missing
  (however, cf. Delbrück scattering



but in high ω-regime)

- lbl contributes as a subprocess ~~to~~ $g-2$

(e.g. hadronic lbl still insufficiently understood in $g-2$ of the muon)

4. Inhomogeneous fields: Worldline formalism

Euclidean space, Scalar QED

Let us go back to

$$\int d^4x \, \mathcal{L}^1_{HE} = -\int_0^\infty \frac{dT}{T} \, e^{-m^2 T} \, \mathrm{Tr} \, e^{-(-D^2)T} \qquad (47)$$

Interpret $-D^2$ as a quantum mechanical Hamiltonian

$$H_W = -D^2$$

for a single particle with time evolution T:

$$\mathcal{U}(T) = e^{-H_W T} \qquad (48)$$

$$\Rightarrow \mathrm{Tr} \, e^{-(-D^2)T} = \mathrm{Tr} \, \mathcal{U}(T)$$

trace of QM time evolution operator can be represented as a path integral

$$\mathrm{Tr} \, e^{-(-D^2)T} = \mathcal{N} \int \mathcal{D}x \, e^{-\int_0^T d\tau \, L_W} \qquad (49)$$

where L_W is the corresponding Lagrangian

Legendre back-transformation of H_W:

$$L_W = \frac{1}{4} \dot{x}^2(\tau) + ie\, \dot{x}_\mu(\tau)\, A_\mu(x(\tau)) \tag{50}$$

$$\Rightarrow \int d^4x\, \mathcal{L}^1_{HE} = -\int_0^\infty \frac{dT}{T}\, e^{-m^2 T}\, \mathcal{N} \int \mathcal{D}x\, e^{-\int_0^T \frac{\dot{x}^2}{4} d\tau}\, e^{-ie\oint dx_\mu A_\mu} \tag{51}$$

Worldline representation of HE-action

- has great advantages in analytic calculations
  (cf. C. Schubert "string-inspired" approach)

- can be evaluated numerically for arbitrary $A_\mu(x)$

3.4 Pair production in electric fields

Vacuum persistence amplitude

$$e^{i\int d^4x\, \mathcal{L}_{HE}} = \langle 0|0\rangle_A \tag{52}$$

$$\Rightarrow \text{ if } \mathcal{L}_{HE} \in \mathbb{R} \quad \Rightarrow \quad |\langle 0|0\rangle_A|^2 = 1 \tag{53}$$

stable vacuum

$$\text{if } \operatorname{Im} \mathcal{L}_{HE} > 0 \quad \Rightarrow \quad 0 < |\langle 0|0\rangle_A|^2 < 1$$

$$P = 1 - |\langle 0|0\rangle_A|^2 = 1 - e^{-2\int d^4x\, \operatorname{Im}\mathcal{L}_{HE}} \tag{54}$$

probability for vacuum $\underline{\underline{\text{decay}}}$

decay rate

$$w = 2 \operatorname{Im} \mathcal{L}_{HE} \tag{55}$$

Heisenberg-Euler Lagrangian in an electric field

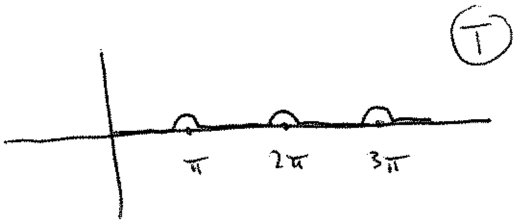
$$\mathcal{L}^{1}_{HE} = \frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3} e^{-m^2T}\left(eET \cot eET - 1 + \frac{1}{3}(eET)^2\right)$$

$$= \frac{(eE)^2}{8\pi^2}\int_0^\infty \frac{dT}{T^3} e^{-\frac{m^2}{eE}T}\left(T\frac{\cos T}{\sin T} - 1 + \frac{1}{3}T^2\right) \qquad (56)$$

$\sin T$ has zeros at $T = n\pi$, $n = 1,2,3,\dots$

We use

$$\frac{1}{\sin T} \xrightarrow{\mathrm{Im}\,\mathcal{L}} i\pi \sum_{n=1}^{\infty} \delta(T - n\pi)\,(-1)^n \qquad (57)$$



Semi-circles give imaginary part (residue theorem)

$$\Rightarrow \quad \underline{\underline{w}} = 2\,\mathrm{Im}\,\mathcal{L}_{HE} = \underline{\underline{\frac{(eE)^2}{4\pi^3}\sum_{n=1}^{\infty}\frac{e^{-\frac{m^2}{eE}n\pi}}{n^2}}} \qquad (58)$$

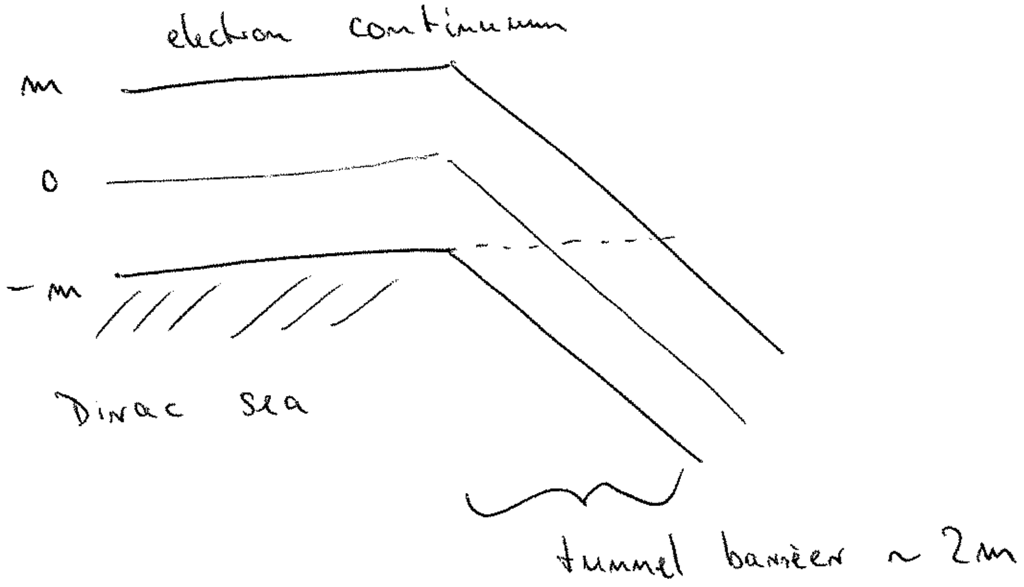
Schwinger vacuum decay rate

Weak-field limit:

$$W\,(eE \ll m^2) \simeq \frac{(eE)^2}{4\pi^3}\, e^{-\pi \frac{m^2}{eE}} \qquad (59)$$

expontially damped!

Similar to quantum mechanical tunneling



pair production is non-perturbative in (eE)

Simple Schwinger calculation • neglects back-reactions (produced pairs screen the field → plasma oscillations), which become relevant for extremely strong fields $E \gtrsim E_{cr}$.

• provides no information about
real-time dynamics
spectrum of produced pairs
} (cf. talk by F. Hebenstreit)