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Semiclassics of Andreev billiards

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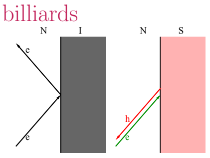
Abstract

By coupling a superconductor to a quantum cavity or dot one may decide whether its classical counterpart has integrable or chaotic dynamics by looking at the density of states. Random matrix theory (RMT) predicts a true gap of the order of the Thoules energy E_T if the classical motion is chaotic and an exponential damping in the case of integrable motion. We use semiclassical techniques developed in the recent years to reproduce the RMT predictions. Using the same techniques we show that the existance of a superconductor in proximity of the dot has crucial effects on the conductance.

Andreev reflection and Andreev billiards



normal reflection   Andreev reflection

In 1964 Andreev found a new scattering mechanism occurring at the interface between a normal and a superconducting region [1]: An electron (hole) in the normal region hitting the superconductor is retro-reflected as a hole (electron) which additionally accumulates a phase $e^{-i\phi}$ ($e^{i\phi}$) where ϕ is the superconducting phase. This scattering mechanism gives rise to a new type of billiards: Andreev billiards consist of a ballistic region inclosed by a boundary partially Andreev reflecting. Experimentally this can be realized by ballistic quantum dots coupled to superconductors.
Quantum mechanically Andreev billiards are described by the Bogoliubov-De Gennes (BEG) equation [2]

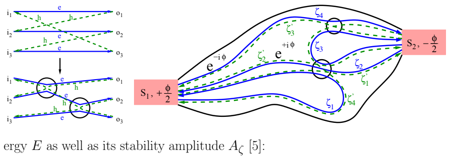
$$\begin{pmatrix} H & \Delta \\ \Delta^* & -H^* \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = E\begin{pmatrix} u \\ v \end{pmatrix}$$

Within the so called scattering approach the average density of states of Andreev billiards with two superconductors with phases $\pm\phi$ normalised to the density of states of the isolated billiard is given by [2, 3, 4]

$$d(\epsilon) = 1 + \frac{2}{N_S}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{Im}\frac{\partial}{\partial\epsilon}\mathrm{Tr}\left[e^{-i\tilde{\phi}}S^*\left(-\frac{\epsilon\hbar}{2\tau_D}\right)e^{i\tilde{\phi}}S\left(\frac{\epsilon\hbar}{2\tau_D}\right)\right]^n, \quad (1)$$

where ϵ is the energy measured with respect to the Fermi energy E_F and in units of the Thouless energy $E_T = \frac{2\hbar}{\tau_D}$ with the mean dwell time τ_D. $N_S = N_{S_1} + N_{S_2}$ is the total number of channels provided by the superconductors, $S(E)$ is the scattering matrix at energy E and $\tilde{\phi}$ is a diagonal matrix with the first N_{S_1} entries being ϕ and the remaining N_{S_2} ones being $-\phi$.

Semiclassical approach

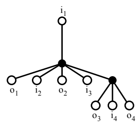


In a trajectory based semiclassical approach the entries of the scattering matrix are determined by classical trajectories ζ connecting the incoming channel i to the outgoing channel j and its classical action $S_\zeta(E)$ at energy E as well as its stability amplitude A_ζ [5]:

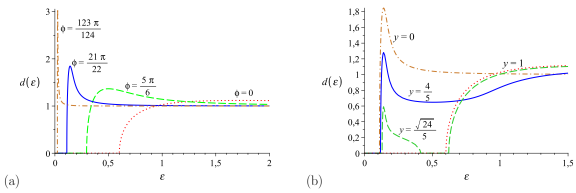
$$S_{oi}(E) = \frac{1}{\sqrt{T_H}}\sum_{\zeta(i\to o)} A_\zeta \exp\left(\frac{i}{\hbar}S_\zeta(E)\right),$$

where T_H is the Heisenberg time corresponding to the mean level spacing.
Inserting this into the expression for the density of states (1) provides a phase $S_{\zeta_m}(\epsilon) - S_{\zeta'_m}(-\epsilon)$ which in the semiclassical limit $\hbar \to 0$ oscillates widely and therefore cancels when averaging over billiard shape unless the classical action differences are of order of $\hbar$.
In order to ensure an action difference that is small enough the unprimed (electron) trajectories have to be collapsed onto the primed (hole) trajectories which leads to the formation of encounters. An l-encounter is a small region where l unprimed trajectories avoid to cross each other in configuration space while l primed ones cross each other. An e-h path pair then contributes a factor $[N(1-i\epsilon)]^{-1}$ while an l-encounter inside the dot contributes a factor $-N(1-il\epsilon)$. A path pair or an l-encounter hitting the superconductor S_i, *i.e.* l path pairs hitting S_i at the same channel, contributes a factor N_{S_i} [6].

Density of states



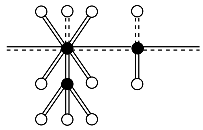
The density of states is related to the correlation functions $C(\epsilon,n,\phi) = (N_S)^{-1}\,\mathrm{Tr}\left[e^{-i\tilde{\phi}}S^*\left(-\frac{\epsilon\hbar}{2\tau_D}\right)e^{i\tilde{\phi}}S\left(\frac{\epsilon\hbar}{2\tau_D}\right)\right]^n$ of n scattering matrices. In leading order in the inverse channel number the correlation functions may be calculated by identifying the set of classical trajectories by rooted plane trees, where an l-encounter is represented by a node of degree $2l$ and the path pairs by straight lines. Each tree is then characterised by a vector $\boldsymbol{v}$ whose l-th entry is the number of l-encounters and satisfies $\sum_{l=2}^{\infty}(l-1)v_l + 1 = n$ [7]. By cutting each tree at the top node such that it decomposes into several subtrees, one finds a recursion relation for each tree. If $N_{S_1} = N_{S_2}$ this yields a fourth order equation for the generating function of the correlation function $G(\epsilon,r,\phi) = \sum_{n=1}^{\infty} C(\epsilon,n,\phi)r^n$ [7] while in general one gets two coupled equations for $G(\epsilon,r,\phi)$ and $G(\epsilon,r,-\phi)$. By solving this equation and integrating over r one finally finds the density of states [8, 4]. In the case that the two superconducting leads both provide the same number of channels the RMT prediction could be reproduced (figure 1a) while a large difference in the numbers of channels causes an intermediate gap. (figure 1b)



(a)   (b)

Figure 1: Density of states of an Andreev billiard with two superconducting leads with (a) equal numbers of channels at different phase differences and with (b) $y = (N_{S_1} - N_{S_2})/N_S$ at phase difference $\phi = 21\pi/22$. If the phase difference is large enough, by increasing the difference between the numbers of channels a intermediate gap may occur.

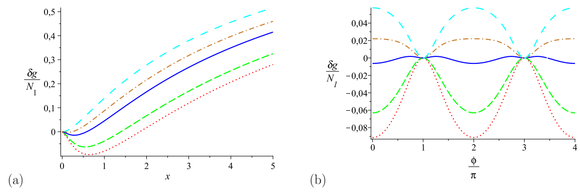
Conductance



The conductance for electrical transport between the leads k and l of an Andreev billiard with two normal leads attached is in the linear response regime at low temperature given by [9]

$$G_{kl} = \frac{e}{\pi\hbar}\left(2N_k\delta_{kl} - T^{ee}_{kl} + T^{eh}_{kl} - T^{hh}_{kl} + T^{he}_{kl}\right),$$

where $T^{\alpha\beta}_{kl} = \sum_{o\in k, i\in l}\left|S^{\alpha\beta}_{oi}\right|$ is the transmission coefficient for transmission from lead l to lead k while converting the β-type quasiparticle in an α-type one, N_k is the number of channels in lead k.
In terms of diagrams (by replacing encounters by nodes and path pairs by straight lines called links) the classical trajectories consist of a diagonal stretch consisting of links and nodes connecting lead i and lead j. From the nodes again trees as those for the density of states emerge. If the chemical potential of the superconducting leads is the same as that of one of the two normal leads, only $G_{11} = \frac{e^2}{\pi\hbar}(g_{cl} + \delta g)$, with $g_{cl} = N_1(N_2 + 2N_S)/(N_1 + N_2 + 2N_S)$ being the classical conductance arising from the diagonal contributions with $\zeta_1 = \zeta_2$ without any encounter and δg being the quantum correction, plays a role.



(a)   (b)

Figure 2: Conductance correction for $N_2/N_1 \to 0$ (dotted), $N_2/N_1 = 0.2$ (dashed), $N_2/N_1 = 1$ (solid), $N_2/N_1 = 2$ (dash dotted) and $N_2/N_1 = 7$ (space dashed) as a function of (a) $x = N_S/(N_1 + N_2)$ at $\phi = 0$ and (b) as a function of the phase difference ϕ at $x = 0.5$.

Conclusion and Outlook

Semiclassics based on the diagonal contribution is not sufficient to explain Andreev billiards. In fact correlations between several orbits are important to describe the effects of superconductors in proximity to a quantum dot.
The results for the conductance show that the effect of the superconductor is of the order of the numbers of channels and may therefore be very important. The phase difference between the superconductors may also play a crucial role.
Note that the results presented here are only valid for large total numbers of channels $N = N_1 + N_2 + N_S$ but each specific number of channels N_1, N_2, N_{S_1} and N_{S_2} may be small. This however means that *e.g.* weak localisation corrections have not been included yet.
With the same approach it is also possible to treat the thermopower of Andreev billiards as already done at leading order in the ratio $N_S/(N_1 + N_2)$ in [10] as well as Andreev billiards with non-zero Ehrenfest time [8, 4]. An applied magnetic field or in the case of the conductance non-zero temperature has also been considered.
Further possible investigations would be the shot noise of Andreev billiards or the effect of tunnel barriers.

References

[1] A. F. Andreev, *Sov. Phys. JETP* **19**, 1228 (1964).
[2] C. W. J. Beenakker, *Lect. Notes. Phys.* **667**, 131 (2005).
[3] E. Doron and U. Smilansky, *Phys. Rev. Lett.* **68**, 1255 (1992).
[4] J. Kuipers, T. Engl, G. Berkolaiko, C. Petitjean, D. Waltner, and K. Richter, arXiv:1004.1327, 2010.
[5] W. H. Miller, *Adv. Chem. Phys* **30**, 77 (1975).
[6] S. Müller, S. Heusler, P. Braun, and F. Haake, *New J. Phys.* **9**, 12 (2007).
[7] G. Berkolaiko, J. M. Harrison, and M. Novaes, *J. Phys. A* **41**, 365102 (2008).
[8] J. Kuipers, D. Waltner, C. Petitjean, G. Berkolaiko, and K. Richter, *Phys. Rev. Lett.* **104**, 027001 (2010).
[9] C. J. Lambert and R. Raimondi, *J. Phys.: Cond. Mat.* **10**, 901 (1998).
[10] P. Jacquod and R. S. Whitney, arXiv:0910.2943, 2009.