Sticky and Non-sticky open Mushrooms

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Introduction

A billiard is a dynamical system in which a particle alternates between motion in a straight line and specular reflections from a boundary. The mushroom billiard forms a class of dynamical systems with sharply divided phase in two dimensions. Its mixed phase space is composed of a single completely regular (integrable) component and a single chaotic and ergodic component. For typical values of the control parameter of the system, an infinite number of marginally unstable periodic orbits (MUPOs) exist making the system sticky in the sense that unstable periodic orbits approach regular regions in phase space and thus exhibit regular behaviour for long periods of time. The problem of finding these MUPOs is expressed as the well known problem of finding optimal rational approximations of a number, subject to some system-specific constraints. We introduce a measure zero set of parameter values for which all MUPOs are destroyed and thus the system is non-sticky. The open mushroom (billiard with a hole) is considered and the asymptotic survival probability function \( P(t) \) is calculated for both cases.

Finding the MUPOs

\[
\cos \frac{\pi}{n} - \frac{r}{2R} < \cos \frac{\pi}{n} \cos \vartheta < \frac{r}{2R},
\]

where \( s \) and \( j \) are positive coprime integers such that \( s \) is the period of the orbit and \( j \) is its rotation number, and \( \lambda \) is 1 if \( s \) is even and 2 if odd. Rearranging and expanding for large \( s \):

\[
\frac{1}{2} s > \vartheta > \frac{1}{2} \left( \frac{\pi \cos \vartheta}{2} \right) + O \left( \frac{1}{s^2} \right).
\]

Results

- Use continued fractions representation: \( \frac{a}{\vartheta} = [a_0, a_1, a_2, \ldots] \).
- If \( K \leq \frac{1}{2} \) and \( |\frac{a}{\vartheta} - \frac{s}{t}\vartheta| < \frac{\vartheta}{2}, \) then \( \frac{a}{\vartheta} \) is always a convergent of \( \frac{s}{t} \vartheta \). Look at convergents \( \frac{a_n}{\vartheta} \).
- \( \frac{a}{\vartheta} - \frac{s}{t} \vartheta = \frac{(-1)^{n-1}}{(a_n + 1)\vartheta^2} \).

Survival Probability

Given a density of particles on the billiard boundary, the probability \( P(t) \) that a particle survives in a strongly chaotic billiard (i.e. does not escape through a small hole of size \( h \)) up to time \( t \) decays exponentially \( \sim e^{-\vartheta t} \). However, in the case of the mushroom there is also an integrable island of orbits in phase space which never escapes, a set of orbits called near-bouncing ball orbits which decay algebraically and a set of orbits near the MUPOs (described above) also decaying algebraically. This is summarised below:

\[
P(t) = \begin{cases} \text{irregular,} & \text{if } t > t^* \medskip \\
\exp(-\vartheta t) + A_0 + B_0 + C_{\text{MUPOs}} + D, & \text{if } t < t^*. \end{cases}
\]

We have obtained leading order expressions for all the above mentioned constants, in particular:

\[
C_{\text{MUPOs}} = \sum_{s,j} \lambda(s+j) \Delta_{s,j},
\]

where

\[
\Delta_{s,j} = \frac{8R \cos^2 \theta_{s,j} \left( \pi - 8s \alpha \cos \left( \frac{\theta_{\text{min}}}{2} \right) \right)^2}{2s^3 \pi^3 |\Delta_{s,j}|^2},
\]

and if the mushroom is non-sticky however, \( S = \emptyset \) and hence \( C_{\text{MUPOs}} = 0 \).

Conclusion

In conclusion, whether the mushroom billiard is sticky or not affects the overall classical dynamics of the system. Therefore, one may expect a quantum mechanical manifestation of this to be observed in mesoscopic and wave physics. Also, it is interesting in general to consider variable Diophantine conditions (i.e. \( K \) as a function of \( \vartheta \)).

References


What are MUPOs

- The mushroom has infinitely many periodic orbits (POs) living in its hat.
- Each PO forms a star polygon in coordinate space.
- Marginally Unstable Periodic Orbits (MUPOs) are PO which cross the central circle of radius \( r \) with \( \vartheta \), Sticky Mushroom.

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references: