Statistics of resonance states in a weakly open chaotic microwave cavity

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CNRS UMR 6622 & Université de Nice-Sophia Antipolis

in collaboration with

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Various experimental open systems

Optical microcavities

Electromagnetic
Reverberation Chambers

Concert hall

Coupling to the environment has to be taken into account to understand realistic systems.

Modeling open systems

Effective Hamiltonian formalism

Losses are modelled
by fictitious open channels

V_{nj} couples the n th resonance to the j th channel

$$H_{eff} = H - \frac{i}{2} V V^\dagger$$

H Hamiltonian of the closed system
matrix of size $(N \times N)$

V Coupling matrix $(N \times M)$

Diagonalization:

$$\mathcal{H}_{eff} |\psi_n\rangle = (E_n - (i/2)\Gamma_n) |\psi_n\rangle$$

Outline

Statistics of open chaotic wave systems

- Statistics of spectral widths (a brief reminder)

- Properties of the complex field

 - Petermann factor, Phase rigidity and Complexness parameter

Description of the statistical models

- The 2-level model

- The N-level model

Statistics of the complexness parameter for the N-level

- Average Complexness parameter and widths fluctuations

- Distribution function for GOE

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Statistics of the spectral widths

Statistics of the spectral widths are well-known from the regime of isolated resonances to the overlapping regime.

Regime of isolated resonances $\langle \Gamma \rangle \ll \Delta$

Using Gaussian coupling amplitudes: The spectral widths are χ^2 distributed

Regime of overlapping resonances $\langle \Gamma \rangle \gtrsim \Delta$

Theoretical study

*H. -J. Sommers, Y. Fyodorov and M. Titov, J. Phys. A. **32**, L77 (1999)*

Experimental study

*U. Kuhl, R. Höhmann, J. Main and H.-J. Stöckmann, Phys. Rev. Lett. **100**, 254101 (2008)*

$f \in [4; 5]$ MHz

$f \in [14.7; 15.7]$ MHz

Properties of the eigenfunctions

CLOSED CHAOTIC SYSTEM with TRS

The system is described by a real Hermitian Hamiltonian

eigenfunctions are real
and orthogonal:

$$\int \psi_n \psi_m = \delta_{nm}$$

The field may be viewed as a random
Gaussian variable (M.V. Berry J. Phys A: Math. Gen. **10**, 2083 (1977))

Direct observation of Berry's hypothesis in a chaotic optical fiber:
Doya et al. PRE **65**, 056223 (2002).

Properties of the eigenfunctions

OPEN CHAOTIC SYSTEM

The system is described by a non Hermitian Hamiltonian

The eigenfunctions are complex: $\psi_n = \psi_n^R + i\psi_n^I$

and bi-orthogonal:

$$\int \left[\psi_n^R \psi_m^R - \psi_n^I \psi_m^I \right] = \delta_{nm} \quad \int \left[\psi_n^R \psi_m^I + \psi_n^I \psi_m^R \right] = 0$$

Real part of the field

Imaginary part of the field

ψ_n^R ψ_n^I may be viewed as independent random Gaussian variables (R. Pnini and B. Shapiro, Phys. Rev. E, **54** (1996) R1032.)

Properties of the complex field

The Petermann Factor K

*K. Petermann, I.E.E.E. **QE-15**, 566 (1979)*

*A. E. Siegman, Phys. Rev. A **39**, 1253 (1989)*

$$K = \left| \frac{\int d\vec{r} [\psi_R^2 + \psi_I^2]}{\int d\vec{r} [\psi_R^2 - \psi_I^2]} \right|^2$$

LINE WIDTH OF A LASER

Schalow-Townes line width

$$\delta\omega_{ST} = \hbar\omega \frac{\Gamma^2}{2P_{out}} \quad \begin{array}{l} \Gamma \text{ spectral width of the passive cavity} \\ P_{out} \text{ output power} \end{array}$$

True fundamental limit of line width

$$\delta\omega \propto K \delta\omega_{ST}$$

The enhancement factor called the **Petermann factor** characterizes the **non orthogonality** of the resonance modes

Properties of the complex field

The phase rigidity ρ^2

P. Brouwer, P.R.E 68, 246205 (2003)

Statistics of wavefunctions in
open chaotic billiard

$$\rho = \frac{\int d\vec{r} [\psi_R^2 - \psi_I^2]}{\int d\vec{r} [\psi_R^2 + \psi_I^2]}$$

Y.-H Kim . et al., P.R.L. 94, 036804, (2005)

Measurement of Long-Range Wave-Function Correlations in an Open Microwave Billiard

The phase rigidity was generally studied as
a continuous function of the frequency

Properties of the complex field

The complexness parameter :

*O. Lobkis and R. Weaver, JASA **108**, 1480 (2000)*
Complex modal statistics in a reverberant dissipative body

$$q_n^2 = \frac{\int d\vec{r} (\psi_n^I)^2}{\int d\vec{r} (\psi_n^R)^2}$$

*J. Barthélemy et. al., Euro. Phys. Lett. **70**, 162 (2005)*

Inhomogeneous resonance broadening and statistics
of complex functions in a chaotic microwave cavity

Through a statistical
ray model:

$$q_n = \frac{\pi S}{cP} \Gamma_n$$

Relations between Phase rigidity, Petermann factor and Complexness parameter

For a **given resonance** state, the quantities are closely related

Using the complexness parameter: $q_n^2 = \frac{\int d\vec{r} (\psi_n^I)^2}{\int d\vec{r} (\psi_n^R)^2}$

$$K_n = \left(\frac{1 + q_n^2}{1 - q_n^2} \right)^2$$

$$\rho_n^2 = \left(\frac{1 - q_n^2}{1 + q_n^2} \right)^2$$

The eigenfunction statistics are investigated
via the complexness parameter

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The 2-level model

*D. Savin, et. al. Europhys. Lett. **76**, 774 (2006)*

Inhomogeneous losses and complexness of wave functions in chaotic cavities

Hypothesis: The complexness parameter is a local quantity

$$H_{eff} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} - \frac{i}{2} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} \quad \text{with} \quad \Gamma_{np} = \sum_{c=1}^M V_n^c V_p^c$$

In the **weak coupling** regime: $\langle \Gamma \rangle = M\sigma^2 \ll \Delta$

$$\text{where } \langle (V_n^c)^2 \rangle = \sigma^2$$

The complexness parameter of both resonances reads:

$$q^2 = \frac{\Gamma_{12}^2}{4(E_2 - E_1)^2}$$

Complexness of elastic modes of a chaotic silicon wafer

*O. Xeridat, C. Poli, O. Legrand, F. Mortessagne, and P. Sebbah, PRE **80** (2009)*

$$p(\varphi) = \frac{q}{2\pi} \frac{1}{q^2 \cos^2 \varphi + \sin^2 \varphi}$$

$$q_n = \frac{\sqrt{(\sum_c V_1^c V_2^c)^2}}{\sum_c (V_n^c)^2} \frac{\Gamma_n}{2(E_2 - E_1)}$$

The N-level model

Effective Hamiltonian in the eigenbasis of H:

$$H_{eff} = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & E_N \end{pmatrix} - \frac{i}{2} \begin{pmatrix} \Gamma_{11} & \cdots & \Gamma_{1N} \\ \vdots & \ddots & \vdots \\ \Gamma_{N1} & \cdots & \Gamma_{NN} \end{pmatrix} \quad \Gamma_{np} = \sum_{c=1}^M V_n^c V_p^c$$

Non-overlapping regime

Eigenvalues:

$$\mathcal{E}_n = E_n - \frac{i}{2} \Gamma_n \quad \text{with} \quad \Gamma_n = \sum_{c=1}^M (V_n^c)^2$$

$\{E_n\}$ statistics correspond to the closed case

$\{\Gamma_n\}$ are χ^2 distributed

The N-level model (2)

Eigenvectors: $|\psi_n\rangle = |n\rangle - i \sum_{p \neq n} \frac{\Gamma_{np}}{2(E_n - E_p)} |p\rangle$

with $\Gamma_{np} = \sum_{c=1}^M V_n^c V_p^c \rightarrow$ interference of resonance states due to common decay channels

where $\{|n\rangle\}$ is the eigenbasis of the closed part

Expression of the **complexness parameter**:

$$q_n^2 = \sum_{p \neq n} \frac{\Gamma_{np}^2}{4(E_n - E_p)^2}$$

Sum of **correlated** random variables

Statistical ensembles of H :

The Gaussian Orthogonal Ensemble

Statistics of chaotic closed systems with TRS are described by GOE:

$$p(H) \propto \exp \left(- \frac{N\pi^2}{4} \text{Tr} H^2 \right)$$

EIGENERGIES

$$p(E_1, \dots, E_N) \propto \prod_{n>m} |E_n - E_m| \exp \left(- \frac{N\pi^2}{4} \sum_n E_n^2 \right)$$

linear level repulsion

The spectrum of H exhibits fluctuations and rigidity

EIGENVECTORS

The $\{\psi_n\}$ are Gaussian distributed

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Average Complexness parameter and widths fluctuations (1)

Rescaled parameters and their statistics

Dimensionless spectral width γ_n

$$\gamma_n = \frac{\Gamma_n}{\sigma^2}$$

χ^2 - distributed :

$$\langle \gamma \rangle = M \quad \text{and} \quad \text{var}(\gamma) = 2M = \frac{2}{M} \langle \gamma \rangle^2$$

$$\text{Distribution: } P_M(\gamma) = \frac{1}{2^{M/2} \Gamma(M/2)} \gamma^{M/2-1} e^{-\gamma/2}$$

Rescaled complexness parameter X_n

$$X_n = \frac{\Delta^2}{\sigma^4} q_n^2$$

$$0 \leq q_n^2 \leq 1 \quad \text{then} \quad 0 \leq X_n \leq \frac{\Delta^2}{\sigma^4}$$

$$\text{Weak coupling regime: } \frac{\Delta^2}{\sigma^4} \gg 1$$

Sokolov's approach

V.V. Sokolov, et. V.G. Zelevinsky Nucl. Phys. **A504**, 562 (1989)

Dynamics and statistics of unstable quantum states

$$\Gamma_{np} = \sum_{c=1}^M V_n^c V_p^c$$

viewed as a scalar product of M-dimensional vectors

$$\Gamma_{np} = \sqrt{\Gamma_n \Gamma_p} \cos \theta_{np}$$

The angle is distributed for $M > 1$ as:

$$p_M(\theta) = \frac{\Gamma(M/2)}{\sqrt{\pi} \Gamma((M-1)/2)} \sin^{M-2} \theta$$

Average Complexness parameter and width fluctuations (2)

$$q_n^2 = \sum_{p \neq n} \frac{\Gamma_{np}^2}{4(E_n - E_p)^2} \text{ yields } X_n = \gamma_n \sum_{p \neq n} \frac{\Delta^2 Z_p}{4(E_n - E_p)^2}$$

where: $Z_p = \gamma_p \cos^2 \theta_{np}$

$$E_n - E_{n+k} = k\Delta$$

Integrating over γ and θ in the definition $P(Z) = \langle \delta(Z - \gamma \cos^2 \theta) \rangle$ yields:

$$P(Z) = \frac{1}{\sqrt{2\pi Z}} e^{-Z/2}$$

independent of M

Since $\langle \gamma \rangle = M$ and $\langle Z \rangle = 1 = \langle \gamma \rangle \langle \cos^2 \theta \rangle$

one obtains: $\langle X \rangle = M f$ where $f = \left\langle \sum_{p \neq n} \frac{\Delta^2}{4(E_n - E_p)^2} \right\rangle$

and also:

$$\langle X \rangle = \frac{f}{2} \text{var}(\gamma) = \frac{f}{M} \langle \gamma \rangle^2$$

Distribution function of X for GOE

$$\langle q^2 \rangle = M f \frac{\sigma^4}{\Delta^2} \quad \text{with} \quad f = \left\langle \sum_{p \neq n} \frac{\Delta^2}{4(E_n - E_p)^2} \right\rangle_n$$

considering the continuous limit: $f = \frac{1}{2} \int_0^\infty dx x^{-2} R_2(x)$

2-point correlation function at small distance: $R_2(x) \sim x$

The average of the complexness factor has
a logarithmic divergence

Summation of the perturbation series is required

But... for all practical purposes, regularized by:

$$f = \frac{1}{2} \int_{\epsilon}^{\infty} dx x^{-2} R_2(x)$$

Probability distribution of the complexness parameter for GOE

$$\mathcal{P}_M^{\text{goe}}(X) = \frac{\pi^2 M}{24X^2} \frac{1 + \frac{\pi^2(3+M)}{4X}}{\left(1 + \frac{\pi^2}{4X}\right)^{M/2+2}}$$

*C. Poli, D. Savin, O. Legrand and F. Mortessagne PRE **80**, 046203 (2009)*

$M = 1$

$M = 5$

$M = 10$

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2D open Chaotic Microwave Cavity

TM polarized electromagnetic field

$$\psi(\mathbf{r}) = E_z(\mathbf{r})$$

solution of

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(\mathbf{r}) = \frac{\tilde{\omega}^2}{c^2}\psi(\mathbf{r})$$

$\tilde{\omega}$
complex angular
frequency

subject to non-Hermitian B.C.s (Ohmic losses)

finite conductivity imposed on
a length l_{abs} of the contour

sliding half-disk

Real (left) and Imaginary (right) components of the 500th resonance state

solved by FEM with Comsol™

Complex eigenvalues

$$\tilde{\omega}_n = \omega_n - i\zeta_n/2$$

Usual correspondance

$$(E_n, \Gamma_n) \text{ with } (\omega_n^2, \omega_n \zeta_n)$$

Numerical results

- Intervals of 100 consecutive resonances above the 300th
- 21 different positions of the movable half-disk

The number of channels :

$$M = \frac{l_{abs}}{\lambda/2}$$

To explore various numbers of channels,

2 frequency ranges :

- from the 300th to the 400th resonances ($\Delta M/M \simeq 14\%$)
- from the 700th to the 800th resonances ($\Delta M/M \simeq 7\%$)

3 different lengths :

$$l_{abs} = \pi/18, \pi/6, \pi/2$$

Weak coupling regime : $\sqrt{\text{var}(\Gamma)}/\Delta < 10^{-2}$

Distribution of $z = \gamma / \langle \gamma \rangle$

$$M = \frac{l_{abs}}{\lambda/2} \simeq \frac{2}{\langle \gamma^2 \rangle / \langle \gamma \rangle^2 - 1}$$

χ^2 -distribution : solid line

No adjustable parameter !

Numerical distribution of the complex parameter

(a) resonances 300 to 400.

(b) resonances 300 to 400.

(c) resonances 700 to 800.

(d) resonances 700 to 800.

$\mathcal{P}_M^{GOE}(X)$: solid line

Summary

- Complexness of eigenfunctions was studied using the effective Hamiltonian formalism & RMT
- Proportionality between the average complexness parameter and the variance of the resonance width
- Exact probability distribution of the complexness parameter derived in the GOE case
- Spatially continuously distributed losses in a chaotic cavity described by a Random Matrix model with a finite number M of coupling channels, which constitutes a variable parameter in the cavity.