Statistics of resonance states in a weakly open chaotic microwave cavity

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in collaboration with
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Various experimental open systems

Coupling to the environment has to be taken into account to understand realistic systems.
Modeling open systems

Effective Hamiltonian formalism

Losses are modelled by fictitious open channels

\( V_{nj} \) couples the \( n \)th resonance to the \( j \)th channel

\[
H_{\text{eff}} = H - \frac{i}{2} V V^\dagger
\]

\( H \) Hamiltonian of the closed system

\( V \) Coupling matrix \((N \times M)\)

Diagonalization:

\[
H_{\text{eff}} |\psi_n\rangle = (E_n - (i/2)\Gamma_n) |\psi_n\rangle
\]
Outline

Statistics of open chaotic wave systems
  Statistics of spectral widths (a brief reminder)
  Properties of the complex field
    Petermann factor, Phase rigidity and Complexness parameter

Description of the statistical models
  The 2-level model
  The N-level model

Statistics of the complexness parameter for the N-level
  Average Complexness parameter and widths fluctuations
  Distribution function for GOE

Statistics of a 2D chaotic open microwave cavity
  FEM numerical results with Ohmic losses at the boundary
  Distribution of the complexness parameter
Statistics of the spectral widths

Statistics of the spectral widths are well-known from the regime of isolated resonances to the overlapping regime.

Regime of isolated resonances $\langle \Gamma \rangle \ll \Delta$

Using Gaussian coupling amplitudes: The spectral widths are $\chi^2$ distributed

Regime of overlapping resonances $\langle \Gamma \rangle \gtrsim \Delta$

Theoretical study


Experimental study


$f \in [4; 5]$ MHz

$f \in [14.7; 15.7]$ MHz
Properties of the eigenfunctions

CLOSED CHAOTIC SYSTEM with TRS

The system is described by a real Hermitian Hamiltonian.

The field may be viewed as a random Gaussian variable (M.V. Berry J. Phys A: Math. Gen. 10, 2083 (1977)).

Direct observation of Berry’s hypothesis in a chaotic optical fiber: Doya et al. PRE 65, 056223 (2002).
Properties of the eigenfunctions

OPEN CHAOTIC SYSTEM

The system is described by a non Hermitian Hamiltonian

The eigenfunctions are complex: \( \psi_n = \psi_n^R + i\psi_n^I \)

and bi-orthogonal:

\[
\int \left[ \psi_n^R \psi_m^R - \psi_n^I \psi_m^I \right] = \delta_{nm} \quad \int \left[ \psi_n^R \psi_m^I + \psi_n^I \psi_m^R \right] = 0
\]

\( \psi_n^R \quad \psi_n^I \) may be viewed as independent random Gaussian variables (R. Pnini and B. Shapiro, Phys. Rev. E, 54 (1996) R1032.)
Properties of the complex field

The Petermann Factor $K$

\[ K = \left| \frac{\int d\vec{r}[\psi_R^2 + \psi_I^2]}{\int d\vec{r}[\psi_R^2 - \psi_I^2]} \right|^2 \]

LINE WIDTH OF A LASER

Schalow-Townes line width

$\delta\omega_{ST} = \hbar\omega \frac{\Gamma^2}{2P_{out}}$  \quad \Gamma \text{ spectral width of the passive cavity}

$P_{out} \text{ output power}$

True fundamental limit of line width

$\delta\omega \propto K\delta\omega_{ST}$

The enhancement factor called the Petermann factor characterizes the non orthogonality of the resonance modes
Properties of the complex field

The phase rigidity $\rho^2$

P. Brouwer, P.R.E 68, 246205 (2003)

Statistics of wavefunctions in open chaotic billiard

\[ \rho = \frac{\int d\vec{r}[\psi_R^2 - \psi_I^2]}{\int d\vec{r}[\psi_R^2 + \psi_I^2]} \]

Measurement of Long-Range Wave-Function Correlations in an Open Microwave Billiard

The phase rigidity was generally studied as a continuous function of the frequency
Properties of the complex field

The complexness parameter:

\[ q_n^2 = \frac{\int d\vec{r}(\psi_I^n)^2}{\int d\vec{r}(\psi_R^n)^2} \]

*O. Lobkis and R. Weaver, JASA 108, 1480 (2000)*
Complex modal statistics in a reverberant dissipative body

*J. Barthélemy et. al., Euro. Phys. Lett. 70, 162 (2005)*

Inhomogeneous resonance broadening and statistics of complex functions in a chaotic microwave cavity

Through a statistical ray model:

\[ q_n = \frac{\pi S}{cP} \Gamma_n \]
Relations between Phase rigidity, Petermann factor and Complexness parameter

For a given resonance state, the quantities are closely related

The eigenfunction statistics are investigated via the complexness parameter

Using the complexness parameter:

\[ q_n^2 = \frac{\int d\vec{r} (\psi_n^I)^2}{\int d\vec{r} (\psi_n^R)^2} \]

\[ K_n = \left( \frac{1 + q_n^2}{1 - q_n^2} \right)^2 \]

\[ \rho_n^2 = \left( \frac{1 - q_n^2}{1 + q_n^2} \right)^2 \]
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Distribution of the complexity parameter
The 2-level model

Hypothesis: The complexity parameter is a local quantity

\[ H_{\text{eff}} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} - \frac{i}{2} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} \quad \text{with} \quad \Gamma_{np} = \sum_{c=1}^{M} V_n^c V_p^c \]

In the weak coupling regime: \( \langle \Gamma \rangle = M \sigma^2 \ll \Delta \)

where \( \langle (V_n^c)^2 \rangle = \sigma^2 \)

The complexity parameter of both resonances reads:

\[ q^2 = \frac{\Gamma_{12}^2}{4(E_2 - E_1)^2} \]
Complexness of elastic modes of a chaotic silicon wafer

O. Xeridat, C. Poli, O. Legrand, F. Mortessagne, and P. Sebbah, PRE 80 (2009)

\[ q_n = \sqrt{\left( \sum_c V_1^c V_2^c \right)^2 / \sum_c (V_n^c)^2} \frac{\Gamma_n}{2(E_2 - E_1)} \]

\[ p(\varphi) = \frac{q}{2\pi q^2 \cos^2 \varphi + \sin^2 \varphi} \]
The N-level model

Effective Hamiltonian in the eigenbasis of $H$:

$$H_{\text{eff}} = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & E_N \end{pmatrix} - \frac{i}{2} \begin{pmatrix} \Gamma_{11} & \cdots & \Gamma_{1N} \\ \vdots & \ddots & \vdots \\ \Gamma_{N1} & \cdots & \Gamma_{NN} \end{pmatrix}$$

Non-overlapping regime

Eigenvalues:

$$\mathcal{E}_n = E_n - \frac{i}{2} \Gamma_n$$

with

$$\Gamma_n = \sum_{c=1}^{M} (V_{nc}^c)^2$$

$$\{E_n\}$$ statistics correspond to the closed case

$$\{\Gamma_n\}$$ are $\chi^2$ distributed
The N-level model (2)

Eigenvectors: \[ |\psi_n\rangle = |n\rangle - i \sum_{p \neq n} \frac{\Gamma_{np}}{2(E_n - E_p)} |p\rangle \]

with \[ \Gamma_{np} = \sum_{c=1}^{M} V_n^{c} V_p^{c} \] → interference of resonance states due to common decay channels

where \{ |n\rangle \} is the eigenbasis of the closed part

Expression of the complexity parameter:

\[ q_n^2 = \sum_{p \neq n} \frac{\Gamma_{np}^2}{4(E_n - E_p)^2} \]

Sum of correlated random variables
Statistical ensembles of $H$: The Gaussian Orthogonal Ensemble

Statistics of chaotic closed systems with TRS are described by GOE:

$$p(H) \propto \exp \left( - \frac{N\pi^2}{4} TrH^2 \right)$$

**EIGENENERGIES**

$$p(E_1, \ldots, E_N) \propto \prod_{n>m} |E_n - E_m| \exp \left( - \frac{N\pi^2}{4} \sum_n E_n^2 \right)$$

linear level repulsion

The spectrum of $H$ exhibits fluctuations and rigidity

**EIGENVECTORS**

The $\{\psi_n\}$ are Gaussian distributed
Statistics of open chaotic wave systems

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Rescaled parameters and their statistics

Average Complexness parameter and widths fluctuations (1)

Rescaled parameters and their statistics

Dimensionless spectral width $\gamma_n$

$\chi^2$-distributed:

$$\langle \gamma \rangle = M \quad \text{and} \quad \text{var}(\gamma) = 2M = \frac{2}{M} \langle \gamma \rangle^2$$

Distribution:

$$P_M(\gamma) = \frac{1}{2^{M/2} \Gamma(M/2)} \gamma^{M/2-1} e^{-\gamma/2}$$

Rescaled complexness parameter $X_n$

$$0 \leq q_n^2 \leq 1 \quad \text{then} \quad 0 \leq X_n \leq \frac{\Delta^2}{\sigma^4}$$

Weak coupling regime:

$$\frac{\Delta^2}{\sigma^4} \gg 1$$
Sokolov’s approach


Dynamics and statistics of unstable quantum states

\[ \Gamma_{np} = \sum_{c=1}^{M} V_n^c V_p^c \]

viewed as a scalar product of M-dimensional vectors

The angle is distributed for \( M > 1 \) as:

\[ p_M(\theta) = \frac{\Gamma(M/2)}{\sqrt{\pi} \Gamma((M-1)/2)} \sin^{M-2} \theta \]
Average Complexness parameter and width fluctuations (2)

\[ q_n^2 = \sum_{p \neq n} \frac{\Gamma_{np}^2}{4(E_n - E_p)^2} \quad \text{yields} \quad X_n = \gamma_n \sum_{p \neq n} \frac{\Delta^2 Z_p}{4(E_n - E_p)^2} \]

where: \( Z_p = \gamma_p \cos^2 \theta_{np} \)

Integrating over \( \gamma \) and \( \theta \) in the definition \( P(Z) = \langle \delta(Z - \gamma \cos^2 \theta) \rangle \)

yields:

\[ P(Z) = \frac{1}{\sqrt{2\pi Z}} e^{-Z/2} \]

independent of \( M \)

Since \( \langle \gamma \rangle = M \) and \( \langle Z \rangle = 1 = \langle \gamma \rangle \langle \cos^2 \theta \rangle \)

one obtains:

\[ \langle X \rangle = M f \]

where \( f = \langle \sum_{p \neq n} \frac{\Delta^2}{4(E_n - E_p)^2} \rangle \)

and also:

\[ \langle X \rangle = \frac{f}{2} \text{var}(\gamma) = \frac{f}{M} \langle \gamma \rangle^2 \]
Distribution function of $X$ for GOE

\[
\langle q^2 \rangle = M f \frac{\sigma^4}{\Delta^2}
\]

with

\[
f = \left\langle \sum_{p \neq n} \frac{\Delta^2}{4(E_n - E_p)^2} \right\rangle_n
\]

considering the continuous limit:

\[
f = \frac{1}{2} \int_0^\infty dx \, x^{-2} R_2(x)
\]

2-point correlation function at small distance: \( R_2(x) \sim x \)

The average of the complexness factor has a logarithmic divergence

Summation of the perturbation series is required

But... for all practical purposes, regularized by:

\[
f = \frac{1}{2} \int_\epsilon^\infty dx \, x^{-2} R_2(x)
\]
Probability distribution of the complexity parameter for GOE

\[ P_{\text{GOE}}^M(X) = \frac{\pi^2 M}{24X^2} \frac{1 + \frac{\pi^2(3+M)}{4X}}{(1 + \frac{\pi^2}{4X})^{M/2+2}} \]

C. Poli, D. Savin, O. Legrand and F. Mortessagne PRE 80, 046203 (2009)

\[ M = 1 \quad \quad M = 5 \quad \quad M = 10 \]
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2D open Chaotic Microwave Cavity

\( \psi(\mathbf{r}) = E_z(\mathbf{r}) \)

solution of

\[
-\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi(\mathbf{r}) = \frac{\tilde{\omega}^2}{c^2} \psi(\mathbf{r})
\]

subject to non-Hermitian B.C.s (Ohmic losses)

finite conductivity imposed on a length \( l_{abs} \) of the contour

---

\( \tilde{\omega} \) complex angular frequency

\( l_{abs} \)
Real (left) and Imaginary (right) components of the 500th resonance state solved by FEM with Comsol™

Complex eigenvalues
\[ \tilde{\omega}_n = \omega_n - i\frac{\zeta_n}{2} \]

Usual correspondance
\( (E_n, \Gamma_n) \) with \( (\omega^2_n, \omega_n \zeta_n) \)
Numerical results

- Intervals of 100 consecutive resonances above the 300th
- 21 different positions of the movable half-disk

The number of channels: \[ M = \frac{l_{\text{abs}}}{\lambda/2} \]

To explore various numbers of channels, 2 frequency ranges:
- from the 300th to the 400th resonances \((\Delta M/M \approx 14\%)\)
- from the 700th to the 800th resonances \((\Delta M/M \approx 7\%)\)

3 different lengths:
\[ l_{\text{abs}} = \pi/18, \pi/6, \pi/2 \]

Weak coupling regime: \(\sqrt{\text{var}(\Gamma)}/\Delta < 10^{-2}\)
Distribution of $z = \gamma / \langle \gamma \rangle$

\[ M = \frac{l_{abs}}{\lambda/2} \sim \frac{2}{\langle \gamma^2 \rangle / \langle \gamma \rangle^2 - 1} \]

\( \chi^2 \) - distribution: solid line

No adjustable parameter!
Numerical distribution of the complex parameter

(a) resonances 300 to 400.

(b) resonances 300 to 400.

(c) resonances 700 to 800.

(d) resonances 700 to 800.

$\mathcal{P}_M^{GOE}(X)$ : solid line
Complexness of eigenfunctions was studied using the effective Hamiltonian formalism & RMT.

Proportionality between the average complexity parameter and the variance of the resonance width.

Exact probability distribution of the complexity parameter derived in the GOE case.

Spatially continuously distributed losses in a chaotic cavity described by a Random Matrix model with a finite number $M$ of coupling channels, which constitutes a variable parameter in the cavity.