Semiclassical aspects of (chaotic) quantum systems

Part 2: Theory of spectral statistics

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Reminder: Klaus Richter's talk

Bohigas-Giannoni-Schmit conjecture

classical chaos $\Rightarrow$ universal **quantum** level fluctuations

e.g.: **spectral form factor**

$$K(\tau) = \frac{1}{d} \int_{-\infty}^{\infty} d\epsilon \left\langle d_{\text{osc}} \left(E + \frac{\epsilon}{2} \right) d_{\text{osc}} \left(E - \frac{\epsilon}{2} \right) \right\rangle_E \exp(i\epsilon\tau T_H/\hbar)$$

for time-reversal invariant chaotic systems:

M. Gutierrez, PhD thesis,
Regensburg (2008)

$$K_{\text{GOE}}(\tau) = \begin{cases} 2\tau - \tau \log(1 + 2\tau) = 2\tau - 2\tau^2 + 2\tau^3 - \frac{8}{3}\tau^4 & \tau < 1 \\ 2 - \tau \ln \frac{2\tau+1}{2\tau-1} & \tau > 1 \end{cases}$$

Why?

$$K_{\text{GOE}}(\tau) = \begin{cases} 2\tau - \tau \log(1 + 2\tau) = 2\tau - 2\tau^2 + 2\tau^3 - \frac{8}{3}\tau^4 + \dots & \tau < 1 \\ 2 - \tau \ln \frac{2\tau+1}{2\tau-1} & \tau > 1 \end{cases}$$

use Gutzwiller trace formula for d_{osc} :

$$K(\tau) \sim \sum_{\gamma, \gamma'} \left\langle A_{\gamma} A_{\gamma'} e^{i(\mathbf{S}_{\gamma} - \mathbf{S}_{\gamma'})/\hbar} \delta \left(\tau - \frac{T_{\gamma} + T_{\gamma'}}{2T_H} \right) \right\rangle_E$$

need pairs of orbits with small action difference:

- Berry (1985): $\gamma' = \gamma$ or time reversed of γ
- Sieber/Richter (2001):

- more complicated orbit correlations for higher orders in τ
- $\tau > 1$: need to improve semiclassical approximation

Phase-space approach

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2-encounter = part of an orbit where two stretches come close or are almost time-reversed

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Phase-space approach

Directions in phase space of chaotic systems:

- **unstable direction:**
deviations along the unstable direction grow like $e^{\lambda t}$
(λ =Lyapunov exponent)
 $\Rightarrow$ **sensitive dependence on initial conditions**
- **stable direction:**
deviations along the stable direction shrink like $e^{-\lambda t}$
- direction of the flow
- direction of increasing energy

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Characterize encounter by **unstable separation u** and **stable separation s** !

Phase-space approach

u and s determine:

- action difference

$$S_\gamma - S_{\gamma'} \sim us$$

- duration of the encounter

$$t_{\text{enc}}(u, s) \sim \frac{1}{\lambda} \ln \frac{c}{|u|} + \frac{1}{\lambda} \ln \frac{c}{|s|}$$

- probability of encounters with given separations

$$w_T(u, s) \sim \frac{T(T - 2t_{\text{enc}}(u, s))}{\Omega t_{\text{enc}}(u, s)}$$

⇒ Contribution to form factor

$$K_{\text{off}} \sim \sum_{\gamma} |A_{\gamma}|^2 \delta(T - T_{\gamma}) \int du \int ds w_T(u, s) e^{ius/\hbar} = -2\tau^2$$

All orders in τ

All orders in τ

Orbits differ in **arbitrarily many encounters**,
where **arbitrarily many stretches** are close or almost time-reversed

All orders in τ

All orders in τ

Sum over all pairs:

- **time-reversal invariant systems:**

$$\Rightarrow K(\tau) = 2\tau - \tau \log(1 + 2\tau) = 2\tau - 2\tau^2 + 2\tau^3 - \frac{8}{3}\tau^4 + \dots \quad (\tau < 1)$$

as for GOE

- **systems without time-reversal invariance:**

only encounters with parallel stretches allowed
contributions sum to zero

$$\Rightarrow K(\tau) = \tau \quad (\tau < 1), \quad \text{as for GUE}$$

$$\tau > 1$$

Correlation function

Idea:

improve semiclassical approximation

build in that $E_n \in \mathbb{R}$

change setting to make this easier

⇒ Consider **correlation function**

$$\begin{aligned} R(\epsilon) &= \left\langle d\left(E + \frac{\epsilon}{2}\right) d\left(E - \frac{\epsilon}{2}\right) \right\rangle \quad (\text{take } \bar{d} = 1) \\ R_{\text{GUE}}(\epsilon) &= \text{Re} \left(-\frac{1}{2\epsilon^2} + \frac{1}{2\epsilon^2} e^{2\pi i \epsilon} \right) \\ R_{\text{GOE}}(\epsilon) &= \text{Re} \left(\sum_{n \geq 2} c_n \left(\frac{1}{\epsilon}\right)^n + \sum_{n \geq 2} d_n \left(\frac{1}{\epsilon}\right)^n e^{2\pi i \epsilon} \right) \end{aligned}$$

non-oscillatory terms $\leftrightarrow \tau < 1$ oscillatory terms $\leftrightarrow \tau > 1$

Generating function

Represent correlation function through **generating function**:

$$Z(\alpha, \beta, \gamma, \delta) = \frac{\Delta(E + \gamma)\Delta(E - \delta)}{\Delta(E + \alpha)\Delta(E - \beta)}$$

$$\Delta(E) = \det(E - H) = \prod_n (E - E_n)$$

$$R(\epsilon) \propto \operatorname{Re} \frac{\partial^2 Z}{\partial \alpha \partial \beta} \Big|_{\alpha=\beta=\gamma=\delta=\epsilon/2}$$

Proof: use $\det^{-1} = \exp(-\operatorname{tr} \ln)$
 $\Rightarrow \frac{\partial}{\partial E} \Delta(E)^{-1} = -\operatorname{tr} \frac{1}{E-H} \Delta(E)^{-1}$

Motivation: can now build in that $\Delta(E) \in \mathbb{R}$ for $E \in \mathbb{R}$

Conventional semiclassics

$$\text{tr} \left(E^+ - \hat{H} \right)^{-1} \sim \underbrace{-i\pi\bar{d}}_{\text{Weyl}} + \underbrace{\text{sum over classical periodic orbits}}_{\text{Gutzwiller}}$$

$$\Rightarrow \Delta(E^+) \sim \exp \left(\int^{E^+} dE' \text{tr} \left(E' - \hat{H} \right)^{-1} \right)$$

$$\sim \underbrace{e^{-i\pi\bar{d}E^+}}_{\text{Weyl}} \times \underbrace{\sum_A F_A e^{iS_A(E^+)/\hbar}}_{\text{sum over sets of classical periodic orbits (pseudo-orbits) } A}$$

"Resummed" semiclassics

But $\Delta(E)$ should be real for real E

$\Rightarrow$ **Resummation (Berry & Keating)**

$$\left(\sum_{\text{over pseudo-orbits } > T_H/2} \right) = \left(\sum_{\text{over pseudo-orbits } < T_H/2} \right)^*$$

$\Rightarrow$ **Riemann-Siegel lookalike**

$$\Delta(E) = e^{-i\pi E} \times \sum_{A(T_A < T_H/2)} F_A e^{iS_A(E^+)/\hbar} + \text{c.c.}$$

Generating function

$$Z = \frac{\Delta(E + \gamma)\Delta(E - \delta)}{\Delta(E + \alpha)\Delta(E - \beta)}$$

$$\sim e^{\pi i(\alpha + \beta - \gamma - \delta)} \times \sum F_A F_B^* F_C F_D^* e^{i((S_A + S_C) - (S_B + S_D))/\hbar} \dots$$

Weyl factor

sum over pseudo-orbits
 A, B, C, D ($T_C, T_D < T_H/2$)

$$+ \{ \gamma \rightarrow -\delta, \delta \rightarrow -\gamma \}$$

- Weyl factors $\rightarrow 1$ or $e^{2\pi i\epsilon}$ as $\alpha, \beta, \gamma, \delta \rightarrow \epsilon/2$ 😊
- need small action differences

$$\Delta S = (S_A + S_C) - (S_B + S_D)$$

$\Rightarrow$ orbit correlations 😊

Contributions

- **Diagonal approximation:**

(A, C) contain the same orbits as (B, D)

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(A, C) contain the same orbits as (B, D)

- **Encounters:**

Result

Compare to nonlinear sigma model (field theoretical implementation of RMT)

- encounters $\leftrightarrow$ vertices
- links $\leftrightarrow$ propagator lines
- diagrams and their contributions coincide

$\Rightarrow$ Spectral statistics agrees with RMT

Conclusions and outlook

Conclusions:

- Chaotic systems have **universal spectral statistics**
- determined by orbit correlations due to **encounters** in phase space
- use **generating functions** and **Riemann-Siegel lookalike** to understand oscillatory terms in $R(\epsilon)$

Outlook:

Orbit correlations also allow to treat

- transport properties
- deviations from universality outside semiclassical limit
- symmetries (see Chris Joyner's poster)