Anderson localization and correlated disorder

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Outlook

- **Anderson localization: White-noise disorder**
- **Kronig-Penney and Anderson models**
- **Localization length**
- **Correlated disorder**
- **Experiments with single-mode waveguides**
- **Bi-layered structures**

1-D Schrödinger equation :

$$-\frac{\hbar^2}{2m}\psi''(x) + U(x)\psi(x) = E\psi(x)$$

Classical wave equation :

$$\psi''(x) + k^2\psi(x) = U(x)\psi(x) \quad \text{with} \quad E = k^2$$

**Main interest is in the structure of eigenstates
and transport properties for a disorder
with long-range correlations in $U(x)$**

Continuous models with white-noise potentials

with (weak) random potential , $\langle U(x) \rangle = 0$; $\langle U^2(x) \rangle = \sigma^2 \ll 1$

The problem is solved exactly:

$$\langle T \rangle = \sqrt{\frac{2}{\pi}} \left(\frac{l_\infty}{L} \right)^{3/2} \exp\left(-\frac{L}{2l_\infty}\right) \int_0^\infty \frac{z^2}{chz} \exp\left(-z^2 \frac{l_\infty}{2L}\right) dz$$

$$\langle T^2 \rangle = \sqrt{\frac{2}{\pi}} \left(\frac{l_\infty}{L} \right)^{3/2} \exp\left(-\frac{L}{2l_\infty}\right) \int_0^\infty \frac{(2z^2 + z sh2z)}{4ch^3 z} \exp\left(-z^2 \frac{l_\infty}{2L}\right) dz$$

Single-parameter scaling: All transport properties depend on one parameter only

$$\xi = l_\infty / L$$

Anderson localization

in completely random potentials all eigenstates are exponentially
localized

$$|\psi(\mathbf{x})| \sim \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_0|}{l_\infty(E)}\right)$$

$l_\infty(E)$ - localization length (in infinite $\mathbf{x}$ - space)

$$\langle \ln T \rangle = -2L / l_\infty \quad !!!$$

For finite samples of size L the transmission coefficient T is

(a) strongly localized states (insulator)

$$\langle T \rangle \sim \exp(-2L / l_\infty) \quad l_\infty \ll L$$

(b) weakly localized states (conductor)

$$\langle T \rangle \approx 1 - 2L / l_\infty \quad l_\infty \gg L$$

Anderson tight-binding model

ε_n - site energies of an electron

$v = 1$ - hopping matrix element

$$v\psi_{n+1} + v\psi_{n-1} = (E + \varepsilon_n)\psi_n$$

Localization length – White-noise disorder

$$\lambda = l_{\infty}^{-1} = \frac{\sigma^2}{8 \sin^2 \mu}$$

$$\sigma^2 = \langle \varepsilon_n^2 \rangle$$

$$2 \cos \mu = E$$

Kronig-Penney models

(a) amplitude disorder --

(b) positional disorder --

Kronig-Penney model

$$\psi''(x) + k^2 \psi(x) = \sum_{n=-\infty}^{\infty} U_n \psi(x_n) \delta(x - x_n)$$

Here U_n -- amplitude of the n -th delta-scatterer at $x = x_n$

Experimental setup, two limit cases:

(a) amplitude disorder -- $x_n = d n$; U_n - random

(b) positional disorder -- $x_n = d n + \xi_n$; ξ_n - random

$$U_n = U = \text{const}$$

Kronig-Penney model: Amplitude disorder

$$\psi_{n+1} + \psi_{n-1} = \left(2 \cos \mu + \frac{U_n}{k} \sin \mu \right) \psi_n$$

where d is spacing between scatterers

$$U_n = \bar{U} + \xi_n \quad \text{and} \quad \bar{U} = \langle U_n \rangle$$

Kronig-Penney model: Localization length

$$\lambda = l_{\infty}^{-1} = \frac{\sigma^2}{8k^2} \frac{\sin^2 \mu}{\sin^2 \gamma}$$

$$E = k^2 \quad k = \omega / c$$

$$2 \cos \mu + \frac{\overline{U}}{k} \sin \mu = 2 \cos \gamma$$

$$\sigma^2 = \langle \xi_n^2 \rangle \ll \overline{U}^2$$

$$\mu = kd$$

γ is the Bloch phase : $\psi_{n+1} = \exp(i\gamma) \psi_n$

λ - Lyapunov exponent

For single-mode waveguide: $k = \sqrt{(\omega / c)^2 - (\pi / d)^2}$

Localization length – correlated disorder

$$\lambda = l_{\infty}^{-1} = \frac{\sigma^2}{8k^2} \frac{\sin^2 \mu}{\sin^2 \gamma} W(2\mu)$$

where

$$W(2\mu) = 1 + 2 \sum_{m=1}^{\infty} K(m) \cos(2\mu m)$$

and $K(m) = \langle \varepsilon_n \varepsilon_{n+m} \rangle / \langle \varepsilon_n^2 \rangle$ is the pair correlator

$$K(m) = \frac{1}{\pi} \int_0^{\pi} W(\mu) \cos(2\mu m) d\mu$$

How to construct the potential from a given pair correlator?

Method:

$$\varepsilon_n = \sigma \sum_{m=-\infty}^{\infty} \beta_m Z_{n+m}$$

where

$$\beta_m = \frac{1}{\pi} \int_0^{\pi} \sqrt{W(\mu)} \cos(2\mu m) d\mu$$

Z_n

are random numbers with the zero mean
and unitary variance

Numerical simulation – Anderson model

$$\Lambda_0 = l_\infty^{-1} / (l_\infty^{-1})_w$$

$$(l_\infty^{-1})_w = \frac{\sigma^2}{8 \sin^2 \mu}$$

$$2 \cos \mu = E$$

Numerical confirmation: Kronig-Penney model

Figure 17: Transmission coefficient vs. energy for data of Fig.16. (a): correlations in the potential correspond to the dependence $W(\mu) = 0$ outside the interval $[\pi/4, 3\pi/4]$ in the first zone; two zones are shown, $0 < k/\pi < 2$, see Fig.16a. (b): the same for the complementary dependence $W(\mu)$ in the first zone only, see Fig.16b. The length of sequence U_n in both cases is $N = 1000$ (after [KI00]).

Single-mode waveguide: Experimental setup

U.Kuhl, F.Izrailev, A.Krokhin, H.-J.Stöckmann, Appl. Phys. Lett., 77 (2000) 633.

Single-mode waveguide: Experimental setup

634

Appl. Phys. Lett., Vol. 77, No. 5, 31 July 2000

$$\nu = 7.5 \div 15.0 \text{ GHz}$$

Transmission coefficient for 2 correlated sequences

Appl. Phys. Lett., Vol. 77, No. 5, 31 July 2000

Kuhl *et al.*

635

1st sequence

2d sequence

N
100

500

Selective reflection - enhancement of localization

Example

$$2 \cos \mu = E \quad 0 < \mu < \pi$$

$$\lambda = l_{\infty}^{-1} = \frac{\sigma^2}{8k^2} \frac{\sin^2 \mu}{\sin^2 \gamma} W(2\mu)$$

The function $W(\mu)$ is normalized :

$$\int_0^{\pi} W(\mu) d\mu = 1$$

$$\Delta = E_2 - E_1$$

For this case, $K(m) = \frac{c^2}{\pi m} [\sin(2m\mu_2) - \sin(2m\mu_1)]$

and

$$l_{\infty}^{-1} = \frac{\pi \sigma^2}{16 \sin^2 \mu [a \cos(E_2 / 2) - a \cos(E_1 / 2)]} \approx \frac{\pi \sigma^2}{16 \Delta \sin^2 \mu}$$

Enhancement of localization - experiment

FIG. 2 (color online). Profile of intrusion of all 100 micrometer screws into the waveguide for uncorrelated and correlated random sequence U_n . The insets show the corresponding correlation function calculated from the micrometer screw depths [Eq. (7)].

U.Kuhl, F.M.Izrailev, A.A.Krokhin, Phys. Rev. Lett., 100 (2008) 126402.

FIG. 3 (color online). Transmission spectrum $|S_{12}(k)|$ versus position n (vertical scale) of the moving antenna is shown for uncorrelated (a) and correlated (b) disorder. The transmission spectrum $|S_{12}|$ through the whole waveguide is shown by black solid line at the top of each panel. For comparison, the transmission through uncorrelated disorder is added on the top of panel (b) by a yellow dotted line.

FIG. 5 (color online). Details of Fig. 4 for wave numbers between 0.50 and 0.80. In this range of wave numbers one can see several enhanced-localized states. Sharp drop of $|S_{22}|$ at $kd/\pi \approx 0.691$ is due to localized states centered at $n \approx 40$.

FIG. 6 (color online). Profiles of two enhanced-localized states at $kd/\pi \approx 0.63$ (a) and 0.69 (b) for the correlated sequence (marked by vertical dotted lines in Figs. 4 and 5. Graph (b) is the state responsible for a sharp decrease of $|S_{22}|$ shown at the top of Fig. 5. Crosses show an exponential decay with a localization length of 10 scatterer spacings.

Bi-layered structures

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PHYSICAL REVIEW LETTERS

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Localization in Correlated Bilayer Structures: From Photonic Crystals to Metamaterials and Semiconductor Superlattices

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In a unified approach, we study the transport properties of periodic-on-average bilayered photonic crystals, metamaterials, and semiconductor superlattices. Our consideration is based on the analytical expression for the localization length derived for the case of weakly fluctuating widths of layers and takes into account possible correlations in disorder. We analyze how the correlations lead to anomalous properties of transport. In particular, we show that for quarter stack layered media specific correlations can result in a ω^2 dependence of the Lyapunov exponent in *all* spectral bands.

The model

In what follows we consider a propagation of monochromatic electromagnetic TE-wave of frequency ω through an infinite array (stack) of two alternating layers (slabs). Every kind of layers is respectively specified by permittivity (dielectric constant) $\varepsilon_{a,b}$, magnetic permeability $\mu_{a,b}$, corresponding refractive index $n_{a,b}$, impedance $Z_{a,b}$ and wave number $k_{a,b}$,

$$n_a = \sqrt{\varepsilon_a} \sqrt{\mu_a}, \quad k_a = \omega n_a / c = \omega \sqrt{\varepsilon_a} \sqrt{\mu_a} / c, \quad Z_a = \sqrt{\mu_a / \varepsilon_a}; \quad (1.1a)$$

$$n_b = \sqrt{\varepsilon_b} \sqrt{\mu_b}, \quad k_b = \omega n_b / c = \omega \sqrt{\varepsilon_b} \sqrt{\mu_b} / c, \quad Z_b = \sqrt{\mu_b / \varepsilon_b}. \quad (1.1b)$$

Within every a or b layer, the electric field of the wave obeys the wave equation with two boundary conditions formulated on the interfaces between slabs,

$$\left(\frac{d^2}{dz^2} + k_{a,b}^2 \right) \psi_{a,b}(z) = 0, \quad (1.2a)$$

$$\psi_a(z_i) = \psi_b(z_i), \quad \mu_a^{-1} \psi'_a(z_i) = \mu_b^{-1} \psi'_b(z_i). \quad (1.2b)$$

Here z -axis is stretched along the array of bi-layers perpendicular to the stratification, z_i stands for the coordinate of an interface.

Random widths of slabs with correlations

A disorder is incorporated in the structure via the random widths of the slabs (the positional disorder),

$$\tilde{a}(n) = a + \varrho_a(n), \quad \tilde{b}(n) = b + \varrho_b(n). \quad (1.3)$$

Here n enumerates the elementary ab -cells, a and b are the average widths of layers and $\varrho_a(n)$, $\varrho_b(n)$ stand for small variations of the widths. In the absence of disorder the array of bi-layers is periodic with the period $d = a + b$. The random sequences $\varrho_a(n)$, $\varrho_b(n)$ are supposed to be statistically homogeneous with zero average, $\langle \varrho_a(n) \rangle = \langle \varrho_b(n) \rangle = 0$, and binary (pair) correlation functions

$$\langle \varrho_a(n) \varrho_a(n') \rangle = \langle \varrho_a^2(n) \rangle K_a(n - n'), \quad (1.4a)$$

$$\langle \varrho_b(n) \varrho_b(n') \rangle = \langle \varrho_b^2(n) \rangle K_b(n - n'), \quad (1.4b)$$

$$\langle \varrho_a(n) \varrho_b(n') \rangle = \langle \varrho_a(n) \varrho_b(n) \rangle K_{ab}(n - n'). \quad (1.4c)$$

Main result – Lyapunov exponent

$$\lambda = \frac{\varpi^2}{8 \sin^2 \gamma} \left[\sigma_a^2 \mathcal{K}_a(2\gamma) \sin^2 \varphi_b + \sigma_b^2 \mathcal{K}_b(2\gamma) \sin^2 \varphi_a - 2\sigma_{ab}^2 \mathcal{K}_{ab}(2\gamma) \sin \varphi_a \sin \varphi_b \cos \gamma \right],$$

where $\sigma_{a,b}^2 = k_{a,b}^2 \langle \varrho_{a,b}^2(n) \rangle$, $\sigma_{ab}^2 = k_a k_b \langle \varrho_a(n) \varrho_b(n) \rangle$, and $\varpi = Z_a/Z_b - Z_b/Z_a$ is the *mismatching factor*.

$$\cos \gamma = \cos \varphi_a \cos \varphi_b - \frac{1}{2} \left(\frac{Z_b}{Z_a} + \frac{Z_a}{Z_b} \right) \sin \varphi_a \sin \varphi_b.$$

Conventional Photonic Layered Media

$$\epsilon_{a,b}; \mu_{a,b}; n_{a,b} > 0$$

Metamaterials

$$\epsilon_a, \mu_a, n_a > 0; \quad \epsilon_b, \mu_b, n_b < 0$$

Electrons

Quarter stack layered medium

$$n_a a = n_b b$$

$$\xi_a(n) = \xi_b(n)$$

$$\xi_a(n) = -\xi_b(n)$$

FIG. 2 (color online). Lyapunov exponent versus frequency for the quarter stack layered medium, for the plus correlations (left) and minus correlations (right); see Eqs. (8) and (10). Here $n_a = 3.47$ (silicon), $d = 1 \mu\text{m}$, $a = 0.224 \mu\text{m}$, $\langle \varrho_a^2(n) \rangle = 3.75 \times 10^{-5} \mu\text{m}^2$, $c = 1.0$, and $\mathcal{K}_a = 1$.

“Minus-correlations”

$$\xi_a(n) = -\xi_b(n)$$

Anderson Localization in Bi-layer Array with Compositional Disorder: Conventional Photonic Crystals versus Metamaterials

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Abstract

The expression for the localization length has been derived for one-dimensional bi-layered structures with a weak random perturbation of the refractive index. Main attention is paid to the comparison between conventional materials and those consisting of mixed right-hand and left-hand materials. It is shown that the localization length has the same form for both cases.

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One-dimensional Anderson localization in certain correlated random potentials

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We study Anderson localization of ultracold atoms in weak one-dimensional speckle potentials using perturbation theory beyond Born approximation. We show the existence of a series of sharp crossovers (effective mobility edges) between energy regions where localization lengths differ by orders of magnitude. We also point out that the correction to the Born term explicitly depends on the sign of the potential. Our results are in agreement with numerical calculations in a regime relevant for experiments. Finally, we analyze our findings in the light of a diagrammatic approach.

Inhomogeneous DNA: Conducting exons and insulating introns

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Parts of DNA sequences known as exons and introns play very different roles in coding and storage of genetic information. Here we show that their conducting properties are also very different. Taking into account long-range correlations among four basic nucleotides that form double-stranded DNA sequence, we calculate electron localization length for exon and intron regions. Analyzing different DNA molecules, we obtain that the exons have narrow bands of extended states, unlike the introns where all the states are well localized. The band of extended states is due to a specific form of the binary correlation function of the sequence of basic DNA nucleotides.