

AHARONOV – BOHM EFFECT

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Outline

- Electromagnetic potentials in classical and quantum mechanics
- Aharonov–Bohm effect, generalities
- Experimental observations of AB effect
- Scattering on AB vortex
- Deflection on AB vortex
- Billiards with a flux line
- Two vortices
- Summary

Electromagnetic fields in classical mechanics

Two vectors: $\vec{\mathbf{E}}$, $\vec{\mathbf{H}}$

Lorentz force (Henrick Lorentz, 1892, J.C. Maxwell, 1861)

$$\vec{\mathbf{F}} = e\vec{\mathbf{E}} + \frac{e}{c}\vec{\mathbf{v}} \times \vec{\mathbf{H}}$$

Maxwell equations (James Cleck Maxwell, 1865)

$$\nabla \times \vec{\mathbf{E}} = -\frac{1}{c} \frac{\partial \vec{\mathbf{H}}}{\partial t}$$

$$\nabla \cdot \vec{\mathbf{E}} = 4\pi\rho$$

$$\nabla \times \vec{\mathbf{H}} = \frac{1}{c} \frac{\partial \vec{\mathbf{E}}}{\partial t} + \frac{4\pi}{c} \vec{\mathbf{j}}$$

$$\nabla \cdot \vec{\mathbf{H}} = 0$$

ρ is the charge density, $\vec{\mathbf{j}} = \rho\vec{\mathbf{v}}$ is the current density

Variational principle

$$\text{Total action : } S = S_f + S_m + S_{mf}$$

$$\text{Matter action : } S_m = -mc \int ds$$

$$\text{Matter - field action : } S_{mf} = -\frac{e}{c} \int \mathbf{A}_k dx^k$$

$$\text{Field action : } S_f = \frac{1}{16\pi c} \int F_{ik} F^{ik} \prod_j dx^j$$

$$i = 0, 1, 2, 3, \quad s = \sqrt{dx_i dx^i}, \quad x^i = (ct, \vec{r}), \quad x_i = (ct, -\vec{r})$$

$$F_{ik} = \frac{\partial \mathbf{A}_k}{\partial x_i} - \frac{\partial \mathbf{A}_i}{\partial x_k}$$

$$\mathbf{A}_k = (\phi, -\vec{\mathbf{A}}), \quad \phi = \text{scalar potential}, \quad \vec{\mathbf{A}} = \text{vector potential}$$

$$\vec{\mathbf{E}} = -\nabla\phi - \frac{1}{c} \frac{\partial \vec{\mathbf{A}}}{\partial t}, \quad \vec{\mathbf{H}} = \nabla \times \vec{\mathbf{A}}$$

Gauge invariance:

$$A_k \longrightarrow A_k + \partial f(x) / \partial x^k \text{ with } \text{arbitrary} \text{ function } f(x)$$

Quantum mechanics

Schrödinger equation (Erwin Schrödinger, 1925)

$$\hat{H}\Psi = E\Psi$$

$$\hat{H} = \frac{1}{2m}(\hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2) + V(\vec{x}), \quad \hat{p}_j = -i\hbar\frac{\partial}{\partial x_j}$$

In electromagnetic field with potentials ϕ and $\vec{\mathbf{A}}$

$$\hat{p}_j \longrightarrow \hat{p}_j - \frac{e}{c}\mathbf{A}_j, \quad V \longrightarrow V + e\phi$$

$$\hat{H} = \frac{1}{2m} \left(\hat{\vec{p}} - \frac{e}{c}\vec{\mathbf{A}} \right)^2 + V(\vec{x}) + e\phi$$

Gauge invariance in quantum mechanics

Let us consider $\phi = 0$, $\vec{A} = \vec{A}(\vec{x})$. Then $\vec{E} = 0$ and $\vec{H} = \text{rot}\vec{A}$
Schrödinger equation

$$\left[-\frac{\hbar^2}{2m} \left(\frac{\partial}{\partial \vec{x}} - \frac{ie}{\hbar c} \vec{A} \right)^2 \right] \Psi(\vec{x}) = E\Psi(\vec{x})$$

Gauge invariance:

$$\vec{A} \longrightarrow \vec{A} + \frac{\partial f(\vec{x})}{\partial \vec{x}}, \quad \Psi(\vec{x}) \longrightarrow e^{i\frac{ef(\vec{x})}{\hbar c}} \Psi(\vec{x})$$

- In classical mechanics only local fields $\vec{E}(\vec{x})$ and $\vec{H}(\vec{x})$ are important
- In quantum mechanics local potentials $\phi(\vec{x})$ and $\vec{A}(\vec{x})$ enter the Hamiltonian

Aharonov – Bohm effect

In quantum mechanics there are possibilities to observe the influence of electromagnetic potentials though a particle never goes through the region where there exist non-zero electromagnetic fields.

Proc. Phys. Soc. **62**, 8-21, 1949

Phys. Rev. **115**, 485-491, 1959

The Refractive Index in Electron Optics and the Principles of Dynamics

By W. EHRENBERG* AND R. E. SIDAY†

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*MS. received 23rd February 1948, and in amended form 28th July 1948;
read 3rd December 1948*

ABSTRACT. In view of mis-statements made in the literature, the origin of the refractive index in electron optics is discussed in some detail, and the uniqueness of an expression previously given is demonstrated. On this basis, some general properties of electron optics are investigated.

A relation between ray direction and wave normal is obtained. Whereas the refractive index is unique in terms of the magnetic vector potential \mathbf{A} , this itself is arbitrary to some extent. It is shown that \mathbf{A} must, for purposes of electron optics, be chosen so as to satisfy Stokes' theorem and that, if it does, no observable effects result from the arbitrariness of \mathbf{A} . An expression for the optical path difference is given in terms of the magnetic flux enclosed. The results are applied to a number of questions, viz. the differential equations for trajectories, the focusing properties of an axially symmetric field and the interference pattern produced by two converging bundles of rays which enclose a magnetic flux.

THE PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

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AUGUST 1, 1959

Significance of Electromagnetic Potentials in the Quantum Theory

Y. AHARONOV AND D. BOHM

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(Received May 28, 1959; revised manuscript received June 16, 1959)

In this paper, we discuss some interesting properties of the electromagnetic potentials in the quantum domain. We shall show that, contrary to the conclusions of classical mechanics, there exist effects of potentials on charged particles, even in the region where all the fields (and therefore the forces on the particles) vanish. We shall then discuss possible experiments to test these conclusions; and, finally, we shall suggest further possible developments in the interpretation of the potentials.

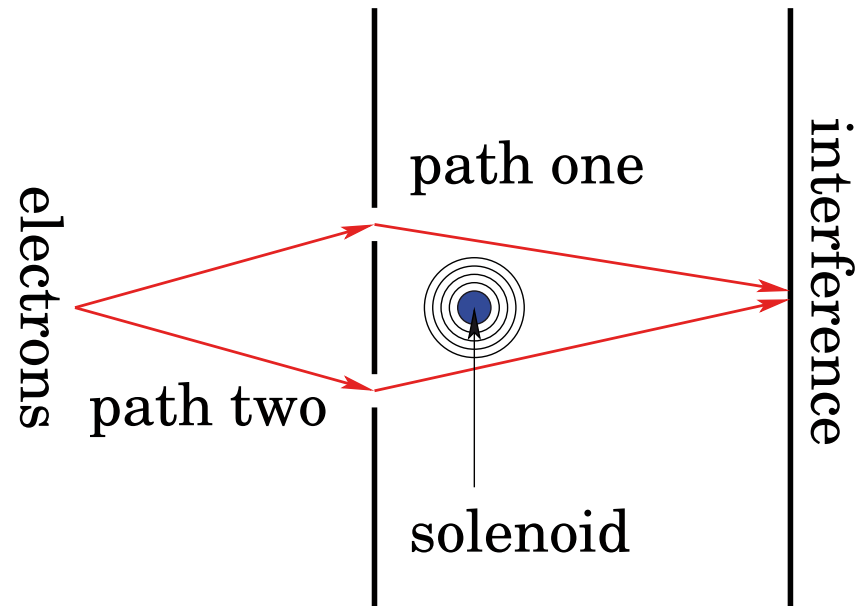


David Bohm
20.12.1917 – 27.10.1992



Yakir Aharonov
28.08.1932

Magnetic Aharonov – Bohm effect

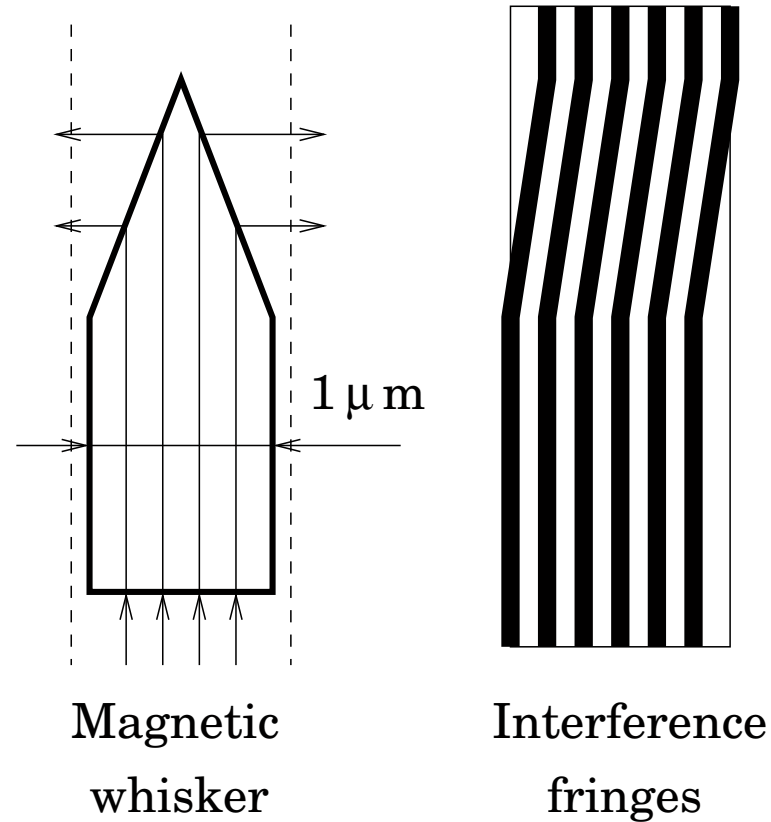


A completely shielded solenoid produces the phase shift

$$\Phi = \frac{e}{\hbar c} \oint \vec{\mathbf{A}} d\vec{r} = \frac{e}{\hbar c} \int \text{rot} \vec{\mathbf{A}} d\vec{s} \equiv \frac{e}{\hbar c} \int \vec{\mathbf{H}} d\vec{s}$$

Shift of an electron interference pattern by enclosed magnetic flux

R. G. Chambers, Phys. Rev. Lett. **5**, 3, 1960

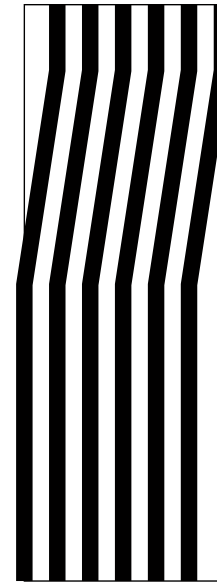


The continuous variation of the phase of electron waves in field free space by means of the magnetic vector potential of a solenoid

G. Möllenstedt, W. Bayh, Physikalische Blätter, 18, 299, 1962



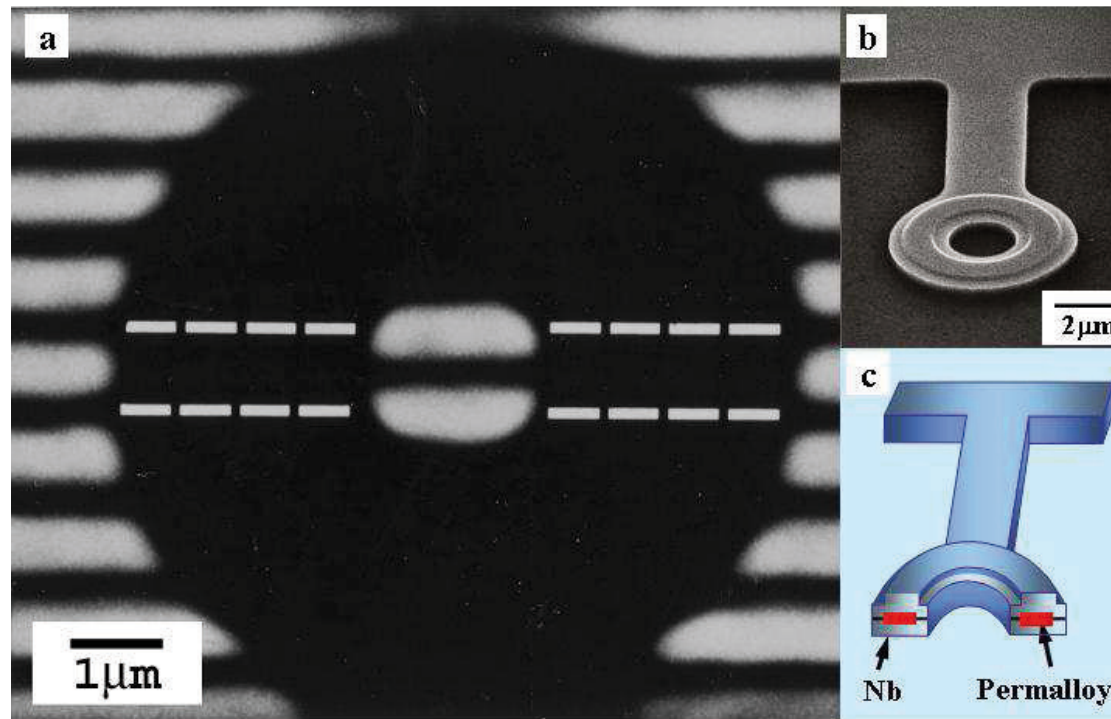
Voltage through the solenoid



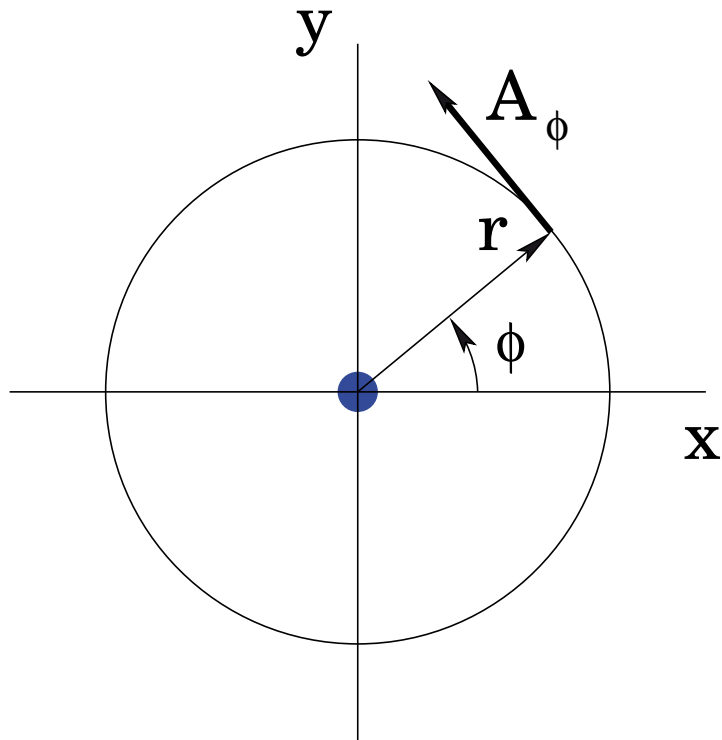
Interference
fringes

Evidence for Aharonov-Bohm effect with magnetic field completely shielded from electron wave

Akiro Tonomura et al., Phys. Rev. Lett. **56**, 792, 1986



Vortex potential



$$A_x = -\frac{\gamma y}{|\vec{r}|^2 + \mathbf{a}^2}$$

$$A_y = \frac{\gamma x}{|\vec{r}|^2 + \mathbf{a}^2}$$

$$\vec{H} = \text{rot} \vec{A}$$

$$H_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \frac{2 \gamma \mathbf{a}^2}{(|\vec{r}|^2 + \mathbf{a}^2)^2}$$

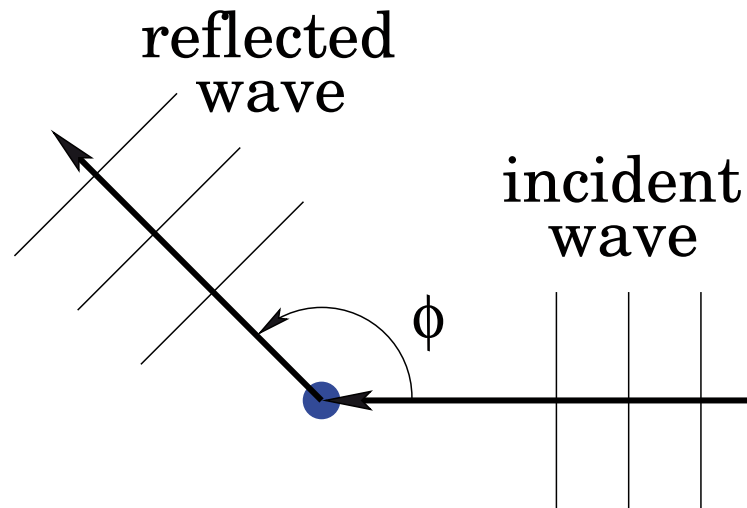
$$H_z \xrightarrow{\mathbf{a} \rightarrow 0} 2\pi\gamma \delta(\vec{x})$$

$$\text{Flux: } \Phi \equiv \oint \vec{H} d\vec{S} = 2\pi\gamma$$

Aharonov-Bohm singular vortex potential

$$A_\phi = \frac{\gamma}{r} = \frac{\Phi}{2\pi r}$$

Scattering on Aharonov-Bohm vortex



Schrödinger equation in dimensionless units

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left(\frac{\partial}{\partial \phi} + i\alpha \right)^2 + k^2 \right] \Psi(r, \phi) = 0$$

$$\alpha = \frac{\Phi}{\Phi_0}, \quad \Phi_0 = -\frac{2\pi\hbar c}{e} = 4.135 \cdot 10^{-7} \text{ gauss cm}^2$$

Elementary solutions

We insist that $\Psi(r, \phi) \rightarrow 0$ when $r \rightarrow$ vortex position ($= 0$).

Elementary solution

$$J_{|m+\alpha|}(kr)e^{im\phi}$$

General solution

$$\Psi(r, \phi) = \sum_{m=-\infty}^{\infty} a_m J_{|m+\alpha|}(kr)e^{im\phi}$$

Initial wave function

Coefficients a_m have to be determined from the condition that Ψ represents a plane wave incident from the right: $\Psi_0(r, \phi) = e^{-ikr \cos \phi}$

But in the end the initial wave function will be different ($|\phi| < \pi$)

$$\Psi_0(r, \phi) = e^{-ikx} e^{-i\alpha\phi} = e^{-ikr \cos \phi - i\alpha\phi}$$

Asymptotic behavior

Expansion

$$e^{-ikr \cos \phi} = \sum_{m=-\infty}^{\infty} (-i)^{|m|} J_{|m|}(kr) e^{im\phi}$$

At $r \rightarrow \infty$, $J_{|m|}(kr) \rightarrow \sqrt{2/(\pi kr)} \cos(kr - \pi|m|/2 - \pi/4)$

At infinity the initial wave has the following asymptotics

$$\Psi_0(r, \phi) = \Psi_{\text{in}}^{(0)}(r, \phi) + \Psi_{\text{out}}^{(0)}(r, \phi)$$

$$\Psi_{\text{in}}^{(0)}(r, \phi) = \left(\frac{1}{2\pi kr} \right)^{1/2} \sum_{m=-\infty}^{\infty} (-i)^{|m|} e^{\pi i|m|/2} e^{-ikr + i\pi/4} e^{im\phi}$$

Incident wave asymptotics for general solution

$$\Psi_{\text{in}}(r, \phi) = \left(\frac{1}{2\pi kr} \right)^{1/2} \sum_{m=-\infty}^{\infty} \mathbf{a}_m e^{i\pi|m+\alpha|/2} e^{-ikr + i\pi/4} e^{im\phi}$$

Exact scattering solution

To have the same incident wave for all m one has to impose

$$a_m = (-i)^{|m+\alpha|}$$

Exact solution

$$\Psi(r, \phi) = \sum_{m=-\infty}^{\infty} (-i)^{|m+\alpha|} J_{|m+\alpha|}(kr) e^{im\phi}$$

Transformations

$$\Psi(r, \phi) = \Psi_1(r, \phi) + \Psi_2(r, \phi) + \Psi_3(r, \phi)$$

$$\Psi_1(r, \phi) = \frac{1}{2} (-i)^\alpha e^{-ikr \cos \phi} \int_0^{kr} e^{ip \cos \phi} \left(J_{\alpha+1}(p) - i J_\alpha(p) e^{i\phi} \right) dp$$

$$\Psi_2(r, \phi, \alpha) = \Psi_1(r, -\phi, -\alpha)$$

$$\Psi_3(r, \phi) = (-i)^{|\alpha|} J_{|\alpha|}(kr)$$

Scattering amplitude

When $r \rightarrow \infty$

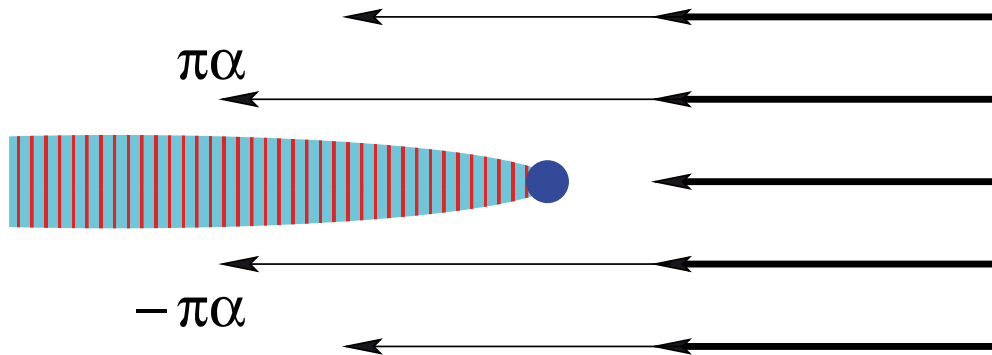
$$\Psi(r, \phi) \longrightarrow e^{-i(kr \cos \phi + \alpha \phi)} + \frac{e^{ikr}}{\sqrt{2\pi ikr}} \mathbf{D}(\phi) e^{-i[\alpha](\phi - \pi)}$$

Diffraction coefficient

$$\mathbf{D}(\phi) = -i \frac{\sin \pi \alpha}{\cos(\phi/2)} e^{-i\phi/2}$$

Scattering cross section

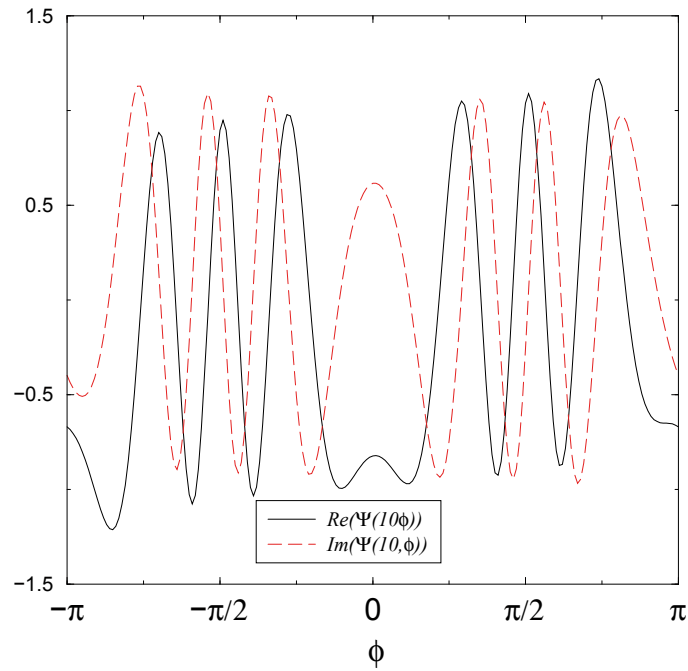
$$\sigma = \frac{\sin^2 \pi \alpha}{2\pi k \cos^2(\phi/2)}$$



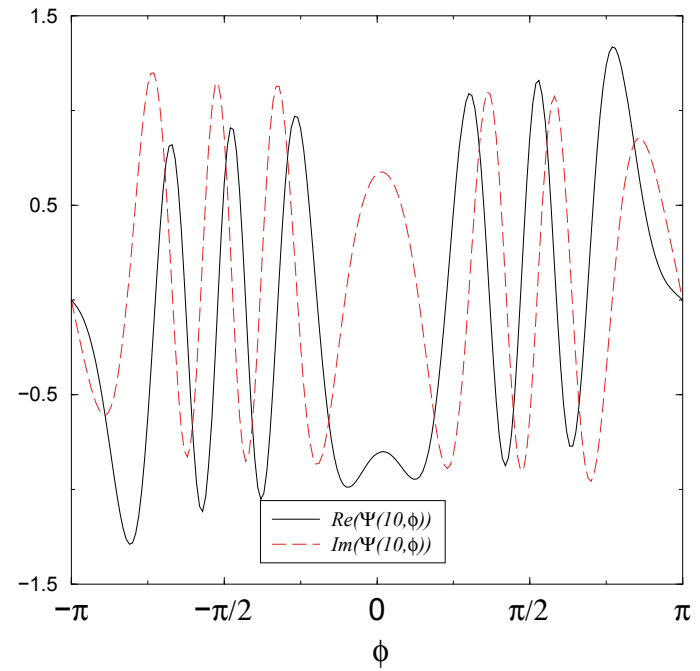
Strong diffraction

Parabolic region $\delta\phi \sim 1/\sqrt{kr}$ in the forward direction

Numerical calculation at $kr = 10$



$\alpha = 0.2$



$\alpha = 0.5$

Uniform approximation at large kr

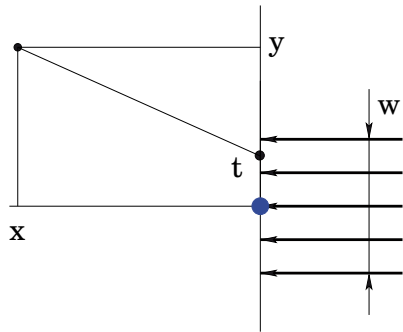
$$\Psi(kr, \phi) \longrightarrow e^{-ikr \cos \phi - i\alpha\phi} - \sin \pi\alpha e^{-i([\alpha](\phi - \pi) + \phi/2 + kr \cos \phi)} \\ + \frac{2 \sin \pi\alpha}{\sqrt{\pi}} e^{-i([\alpha](\phi - \pi) + \phi/2 + kr \cos \phi + \pi/4)} \int_0^{\sqrt{2kr} \cos(\phi/2)} e^{it^2} dt$$

Exact formula for $\alpha = 1/2$

$$\Psi(kr, \phi) = \frac{2}{\sqrt{\pi}} e^{-i(kr \cos \phi + \phi/2 + \pi/4)} \int_0^{\sqrt{2kr} \cos(\phi/2)} e^{it^2} dt$$

Kirchoff approximation at large kr

Kirchoff principle



$$\Psi(x, y) = \frac{k}{\sqrt{2\pi ik|R|}} \int e^{ik\sqrt{x^2+(y-t)^2}} \Psi_{\text{in}}(0, t) dt$$

$$\approx \frac{k}{\sqrt{2\pi ik|x|}} e^{ik|x|} \int e^{ik(y-t)^2/2|x|} \Psi_{\text{in}}(0, t) dt$$

Additional phase close to the axis due to the flux line: $e^{-i\pi\alpha \text{sgn} t}$

Kirchoff (paraxial) approximation

$$\Psi(x, y) \approx \frac{k}{\sqrt{2\pi ik|x|}} e^{-ikx} \int e^{ik(y-t)^2/2|x|} e^{-i\pi\alpha \text{sgn} t} \Psi_{\text{in}}(0, t) dt$$

When $\Psi_{\text{in}}(x, y) = e^{-ikx}$, $\Psi_{\text{in}}(0, t) = 1$ and this results agrees with the above uniform approximation close to the forward direction (ϕ is near π or $-\pi$)

Whirling waves (Berry, 1980)

Poisson summation formula

$$\sum_{m=-\infty}^{\infty} F(m) = \sum_{\mathbf{n}=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\lambda) e^{2\pi i \mathbf{n} \lambda} d\lambda$$

Initial wave

$$e^{-ikr \cos \phi} = \sum_{m=-\infty}^{\infty} e^{-i\pi|m|/2} J_{|m|}(kr) e^{im\phi} = \sum_{\mathbf{n}=-\infty}^{\infty} T_{\mathbf{n}}(r, \phi)$$

$T_{\mathbf{n}}(r, \phi)$ = the \mathbf{n} -th whirling wave

$$\begin{aligned} T_{\mathbf{n}}(r, \phi) &= \int_{-\infty}^{\infty} e^{-i\pi|\lambda|/2} J_{|\lambda|}(kr) e^{i(\phi+2\pi\mathbf{n})\theta\lambda} d\lambda \\ &= e^{-ikr \cos \phi} \delta_{\mathbf{n},0} - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(\pi + iy) e^{ikr \cosh y}}{(\pi + iy)^2 - (\phi + 2\pi\mathbf{n})^2} dy \end{aligned}$$

Interpretation

$T_{\mathbf{n}}(r, \phi)$ = wave at ϕ after \mathbf{n} anticlockwise circuits of the origin.

When the flux line exists, $T_{\mathbf{n}}(r, \phi)$ has to be modified

$$T_{\mathbf{n}}(r, \phi) \longrightarrow T_{\mathbf{n}}(r, \phi)e^{-i\alpha(\phi+2\pi\mathbf{n})}$$

Exact wave function

$$\Psi(r, \phi) = \sum_{\mathbf{n}=-\infty}^{\infty} T_{\mathbf{n}}(r, \phi)e^{-i\alpha(\phi+2\pi\mathbf{n})}$$

$$\Psi(r, \phi) = \sum_{\mathbf{n}=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\pi|\lambda|/2} J_{|\lambda|}(kr) e^{i(\lambda-\alpha)(\phi+2\pi\mathbf{n})} d\lambda$$

Changing $\lambda - \alpha \rightarrow \lambda$ and using the Poisson summation formula gives

$$\Psi(r, \phi) = \sum_{m=-\infty}^{\infty} e^{-i\pi|m+\alpha|/2} J_{|m+\alpha|}(kr) e^{im\phi}$$

Two vortices (Stovicek, 1993)

The S -matrix for scattering on 2 solenoids with fluxes α and β

$$S(\theta, \theta_0) = \sum_{n=0}^{\infty} S^{(n)}(\theta, \theta_0)$$

$$S^{(0)}(\theta, \theta_0) = \frac{1}{2} \{1 + \exp(2\pi i(\alpha + \beta))\} \delta(\theta - \theta_0)$$

$$S^{(n)}(\theta, \theta_0) = \frac{i(-1)^n}{2\pi^n} \sum_{c_n, \dots, c_2, c_1} e^{ic_1 \cos \theta_0 - ic_n \cos \theta}$$

$$\times \int_{R^n} d^n \vec{\tau} e^{\phi_0 \tau_1 + \phi \tau_n} \prod_{j=1}^n \frac{\sin \pi \sigma_j}{\sin \pi(\sigma_j + i\tau_j)} \prod_{j=1}^{n-1} K_{i(\tau_{j+1} - \tau_j)}(-ikr)$$

$c_j = \{a, b\}$, $c_j \neq c_{j+1}$. For $c_j = a$, $\sigma_j = \alpha$, and for $c_j = b$, $\sigma_j = \beta$.
 $\phi_0 = 2\pi - \theta_0$ for $c_1 = a$, $\phi_0 = \pi - \theta_0$ for $c_1 = b$. $\phi = \theta - \pi \pm \pi$ for
 $c_n = a$, $\phi = \theta - \pi$ for $c_n = b$. $K_\nu(x)$ is the Macdonald function

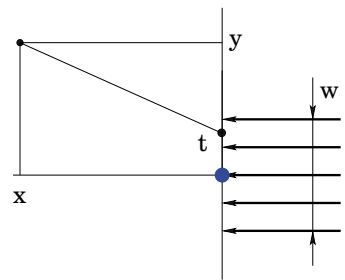
Deflection for Aharonov-Bohm scattering

(Shelankov, 1998)

Cross section for the AB-effect is symmetric $\phi \rightarrow -\phi$

$$\sigma = \frac{\sin^2 \pi \alpha}{2\pi k \cos^2(\phi/2)}$$

One might expect that there is no overall deflection (left-right asymmetry) after scattering. Due to singular forward scattering it is not the case.



Paraxial representation

$$\Psi(x, y) \approx \frac{ke^{ik|x|}}{\sqrt{2\pi ik|x|}} \int e^{ik(y-t)^2/2|x| - i\pi\alpha \text{sgn } t} \Psi_{\text{in}}(0, t) dt$$

Choose a **collimated** beam $\Psi_{\text{in}}(0, t) = e^{-|t|/w}$ and consider $|x| \gg w^2 k$

Fraunhofer diffraction

Cylindrical wave ($R = \sqrt{x^2 + y^2}$, $\phi \approx y/x$)

$$\Psi(x, y) = \frac{1}{\sqrt{2\pi ikR}} e^{ikR} F(\phi)$$

$$F(\phi) = k \int_{-\infty}^{\infty} e^{-ikt\phi - i\pi\alpha \operatorname{sgn} t - |t|/w} dt = 2 \frac{\phi_0 \cos \pi\alpha + \phi \sin \pi\alpha}{\phi^2 + \phi_0^2}$$

$$\phi_0 = 1/(kw)$$

Scattering "cross section"

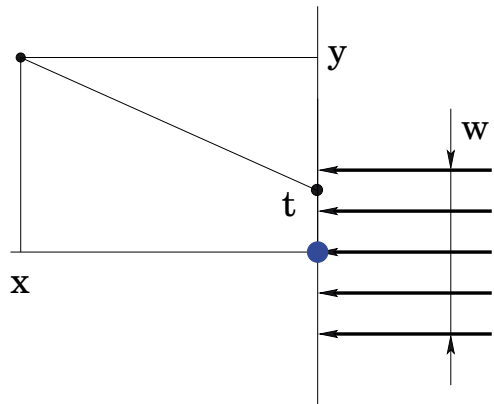
$$\sigma(\phi) \equiv R|\Psi(R, \phi)|^2 = \frac{2(\phi_0 \cos \pi\alpha + \phi \sin \pi\alpha)^2}{\pi k(\phi^2 + \phi_0^2)^2}$$

When $\phi \gg \phi_0$ this expression agrees with the usual AB cross section.

For $\phi \lesssim \phi_0$ it is asymmetric

Mean angle of deflection

$$\Delta\phi = \frac{\int_{-\infty}^{\infty} \phi \sigma(\phi) d\phi}{\int_{-\infty}^{\infty} \sigma(\phi) d\phi} = -\frac{\sin 2\pi\alpha}{kw}$$



In general

$$\Delta\phi \approx -\sin 2\pi\alpha \frac{|\Psi(0)|^2}{k \int_{-\infty}^{\infty} |\Psi(y)|^2 dy}$$

$\Psi(y)$ = wave function without the flux

Effective "Lorentz" force

$$\vec{F} = -\hbar \sin 2\pi\alpha \vec{v} \times \vec{e}_z \xrightarrow{\Phi \rightarrow 0} \frac{e}{c} \vec{v} \times \vec{\Phi}$$

$$\alpha = \Phi/\Phi_0, \Phi_0 = -2\pi\hbar c/e, \Phi = \int H ds$$

S -matrix

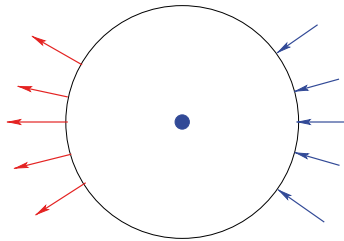
At large distances
the **incoming** wave

$$\Psi_{\text{in}}(r, \phi) = \frac{1}{\sqrt{2\pi kr}} e^{-ikr + i\pi/4} f_{\text{in}}(\phi), \quad f_{\text{in}}(\phi) = \sum_m e^{im\phi} \mathbf{a}_m$$

the **outcoming** wave

$$\Psi_{\text{out}}(r, \phi) = \frac{1}{\sqrt{2\pi kr}} e^{+ikr - i\pi/4} f_{\text{out}}(\phi), \quad f_{\text{out}}(\phi) = \sum_m e^{im\phi} \mathbf{b}_m$$

In general, due to the flux conservation



$$\mathbf{b}_m = \sum_n \mathbf{S}_{mn} \mathbf{a}_n$$

with **unitary** S -matrix

S-matrix for AB scattering

Elementary solution: $J_{|m+\alpha|}(kr)e^{im\phi}$. When $r \rightarrow \infty$

$$\frac{1}{\sqrt{2\pi kr}}e^{-ikr+i\pi/4+i\pi|m+\alpha|/2} + \frac{1}{\sqrt{2\pi kr}}e^{+ikr-i\pi/4-i\pi|m+\alpha|/2}$$

S-matrix: $S_{mn} = \delta_{mn}e^{-i\pi|m+\alpha|}$

$$f_{\text{in}}(\phi) = \sum_m e^{im\phi} \mathbf{a}_m, \quad f_{\text{out}}(\phi) = \sum_m e^{im\phi} \mathbf{b}_m, \quad \boxed{\mathbf{b}_m = e^{-i\pi|m+\alpha|} \mathbf{a}_m}$$

Assuming that the beam is collimated: $\sum_m |a_m|^2 < \infty$

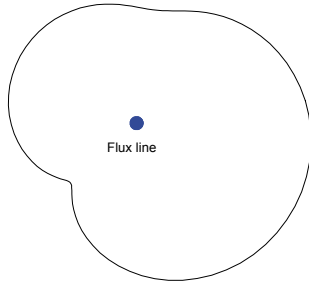
$$f_{\text{out}}(\phi) = \int s_\alpha(\phi - \phi') f_{\text{in}}(\phi') d\phi'$$

$$s_\alpha(\theta) = \cos \pi\alpha \delta(\theta + \pi) - i \frac{\sin \pi\alpha}{2\pi} \text{v.p.} \frac{e^{-i\theta/2}}{\cos(\theta/2)}$$

The interference between two terms gives a term $\sim \sin 2\pi\alpha$ and the mean deflection angle is the same as above.

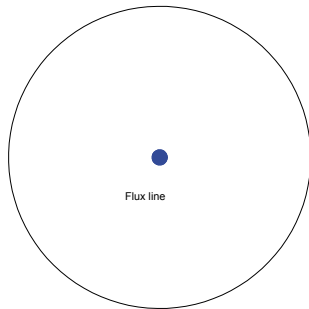
Billiards with a flux line

Chaotic billiards



Introduction of the flux line changes the spectral statistics from GOE to GUE (Berry, Robnik (1986))

Integrable billiard

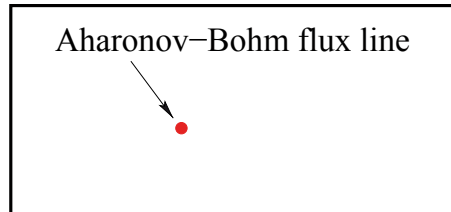


Analysis of the trace formula
Diffractive orbits
(Sieber (1999))

Pseudo-integrable systems

Diffraction orbits

(E.B., Pavloff, Schmit (2000))

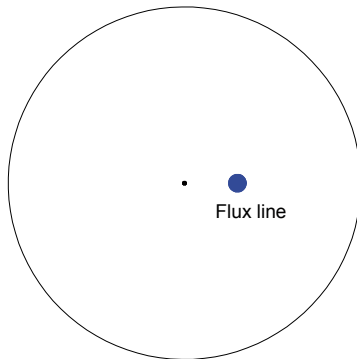


Level compressibility

(E.B., Giraud, Schmit (2002))

$$\chi \equiv K(0) = 1 - 4\alpha(1 - \alpha) + 6\alpha\eta(x_0/a, y_0/b)$$

For irrational rations $\eta = 1/6$



Level compressibility

$$\chi = 1 - 4\lambda\alpha(1 - \alpha)/\pi$$

$$\lambda = \arcsin x + x\sqrt{1 - x^2}, \quad x = d/R$$

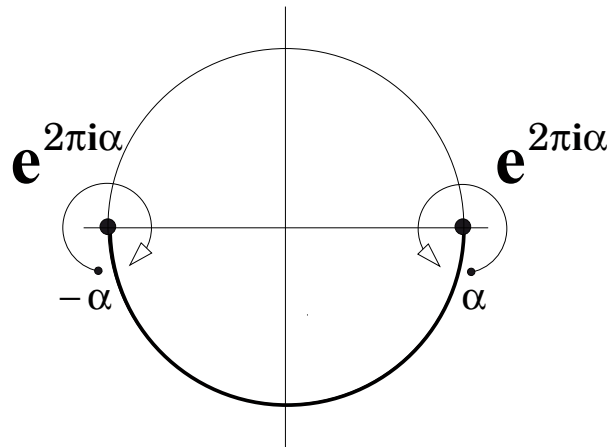
$N(L)$ = number of levels in interval L , normalized that $\bar{N}(L) = L$

Level compressibility $\chi = \langle (N(L) - \bar{N}(L))^2 \rangle$

Scattering on two Aharonov-Bohm vortices with opposite fluxes

(E.B., S. Mashkevich, S. Ouvry (2010))

Singular gauge: $(\Delta + k^2) \Psi = 0$



$$\Psi^{(-)}(R, \phi) = f(\phi) \Psi^{(+)}(R, \phi)$$

$$\frac{\partial}{\partial r} \Psi^{(-)}(R, \phi) = f(\phi) \frac{\partial}{\partial r} \Psi^{(+)}(R, \phi)$$

$f(\phi)$ describes the jump along the cut

$$f(\phi) = \begin{cases} e^{2\pi i \alpha}, & \phi \in [\pi, 2\pi] \\ 1, & \phi \in [0, \pi] \end{cases}$$

Expansions

$$\Psi^{(-)}(kr, \phi) = \sum_{m=-\infty}^{\infty} c_m J_{|m|}(kr) e^{im\phi}$$

$$\Psi_l^{(+)}(kr, \phi) = J_{|l|}(kr) e^{il\phi} + \sum_{n=-\infty}^{\infty} t_n(l) H_{|n|}^{(1)}(kr) e^{in\phi}$$

Equations to fulfill

$$\sum_{n=-\infty}^{\infty} \left(J'_{|m|} H_{|n|}^{(1)} - J_{|m|} H_{|n|}^{(1)'} \right) A_{mn} t_n(l) = -(J'_{|m|} J_{|l|} - J_{|m|} J'_{|l|}) A_{ml}$$

A_{mn} = the Fourier coefficients of $f(\phi)$

$$A_{mn} = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) e^{i(n-m)\phi} d\phi = e^{i\pi\alpha} \begin{cases} \cos \pi\alpha & (m = n) \\ \frac{\sin \pi\alpha}{\pi} \frac{1 - (-1)^{n-m}}{n-m} & (m \neq n) \end{cases}$$

Scattering amplitude

$$\Psi^{(+)}(kr, \phi) \xrightarrow{r \rightarrow \infty} e^{ikr \cos(\theta - \phi)} + \sqrt{\frac{2}{\pi kr}} e^{ikr - i\pi/4} F(\theta, \phi)$$

Standard: $e^{ikr \cos(\theta - \phi)} = \sum_{l=-\infty}^{\infty} i^{|l|} J_{|l|}(kr) e^{il(\phi - \theta)}$

$$\Psi^{(+)}(kr, \phi) = \sum_{l=-\infty}^{\infty} i^{|l|} \Psi_l^{(+)}(kr, \phi) e^{-il\theta} = e^{ikr \cos(\theta - \phi)} + \Psi^{(\text{ref})}(kr, \phi)$$

$\Psi^{(\text{ref})}(kr, \phi)$ is the reflected field

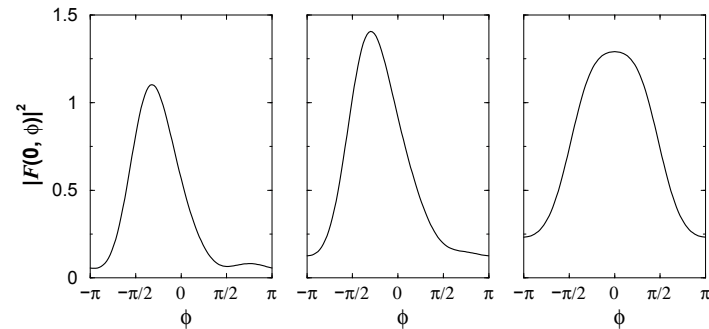
$$\Psi^{(\text{ref})}(kr, \phi) = \sum_{n, l=-\infty}^{\infty} i^{|l|} t_n(l) H_{|n|}^{(1)}(kr) e^{in\phi - il\theta}$$

The scattering amplitude $F(\theta, \phi)$

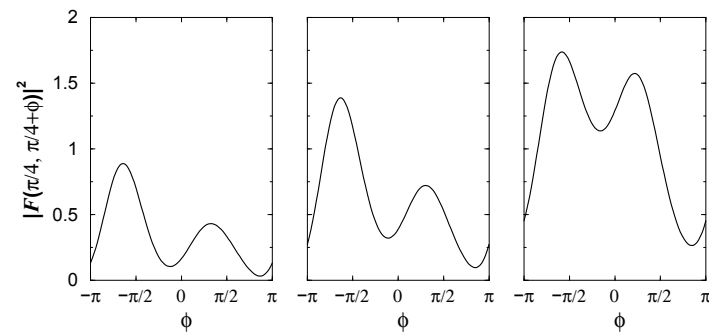
$$F(\theta, \phi) = \sum_{n, l=-\infty}^{\infty} i^{|l| - |n|} t_n(l) e^{in\phi - il\theta}$$

Numerics for $kR = 1$

Left: $\alpha = 1/4$, center: $\alpha = 1/3$, right: $\alpha = 1/2$



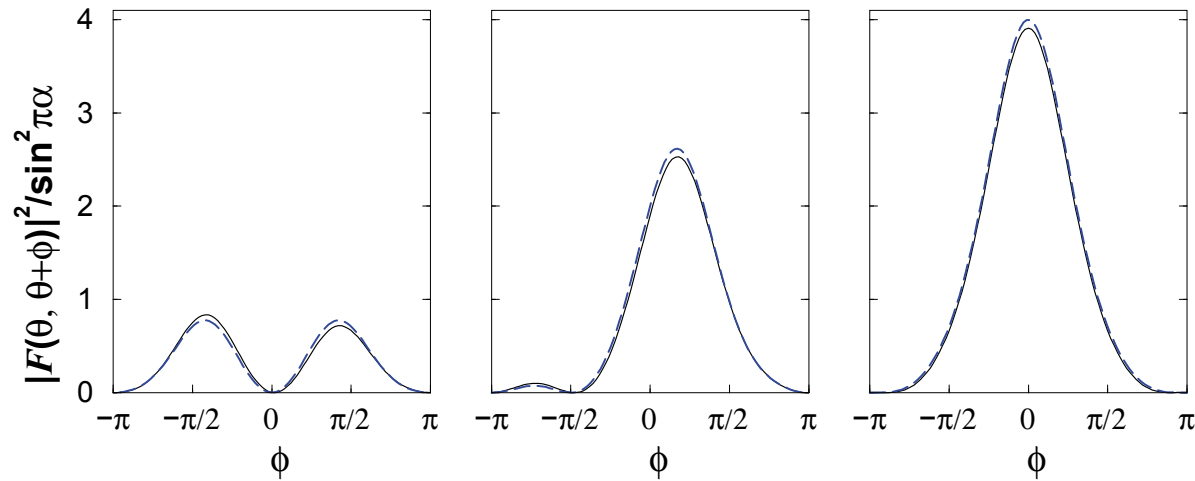
Incident angle $\theta = 0$



Incident angle $\theta = \pi/4$.

Small α limit

$$F(\theta, \theta') \xrightarrow{\sin \pi\alpha \rightarrow 0} -i \sin \pi\alpha \cot \frac{\theta - \theta'}{2} \sin \left(2kR \sin \frac{\theta - \theta'}{2} \sin \frac{\theta + \theta'}{2} \right) .$$



$$\alpha = .01, kR = 1$$

Left: $\theta = 0$, center: $\theta = \pi/4$, right: $\theta = \pi/2$

Small kR limit

For small $x = kR$ it is convenient to introduce

$$t_n(l) = \frac{J_{|l|}(x)}{H_{|n|}^{(1)}(x)} y_n(l)$$

They obey

$$\sum_{n=-\infty}^{\infty} \left(\frac{J'_m}{J_m} - \frac{H_n^{(1)'}}{H_n^{(1)}} \right) A_{mn} y_n(l) = - \left(\frac{J'_m}{J_m} - \frac{J'_l}{J_l} \right) A_{ml}$$

In the limit $x \rightarrow 0$

$$\frac{J'_m(x)}{J_m(x)} \xrightarrow{x \rightarrow 0} \frac{|m|}{x}, \quad \frac{H_n^{(1)'}(x)}{H_n^{(1)}(x)} \xrightarrow{x \rightarrow 0} \frac{-|n| + \delta_{n0}\rho}{x}$$

with $\rho = 2i/(\pi + 2i(\ln(x/2) + \gamma))$.

Final set of equations for unknown $y_n(l)$

$$\sum_{n=-\infty}^{\infty} (|m| + |n| - \delta_{n0}\rho) A_{mn} y_n(l) = -(|m| - |l|) A_{ml}$$

Symmetry $y_{-n}(0) = (-1)^n y_n(0)$.

The scattering amplitude is dominated at small x by S -wave $y_0(0)$

$$F(\theta, \phi) \xrightarrow{x \rightarrow 0} t_0(0) = \frac{y_0(0)}{H_0^{(1)}(x)} \approx -i \frac{\pi}{2} \rho y_0(0)$$

To find $y_0(0)$ it is necessary to solve the infinite system of equations.

In the end

$$F(\theta, \phi) \xrightarrow{kR \rightarrow 0} - \frac{\pi}{\pi + 2i(\ln(kR/2) + \gamma + \beta(\alpha))}$$

$\beta(\alpha)$ is the main quantity of interest.

It is convenient to define for $n \geq 1$ $x_n = 2y_n(0)/(1 + y_0(0))$, $x_0 = 1$,
 $\xi = 2\rho y_0(0)/(1 + y_0(0))$

Final set of equations ($N \rightarrow \infty$)

$$\sum_{n=0}^N A_{mn} x_n = \xi f_m, \quad \frac{\pi}{2} \xi \cot \pi \alpha = \sum_{n=1}^N x_n (1 - (-1)^n)$$

where

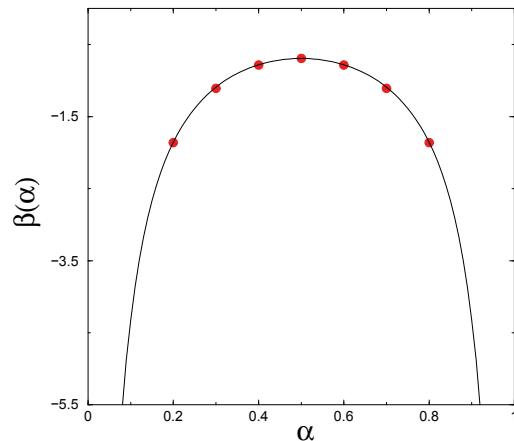
$$f_m = -\frac{\sin \pi \alpha (1 - (-1)^m)}{2\pi m^2} - \frac{\cos \pi \alpha (1 + (-1)^m)}{4m}$$

ξ is a function of α only, $\xi = -2/\beta(\alpha)$.

Numerical determination of $\beta(\alpha)$

Due to power singularities at the vortex locations the convergence with increasing N is slow ($N \approx 500$). General fit

$$\xi \simeq a_0 + \frac{a_1}{N^\alpha} + \frac{a_2}{N^{1-\alpha}} + \frac{a_3}{N} + \frac{a_4}{N^{1+\alpha}}$$

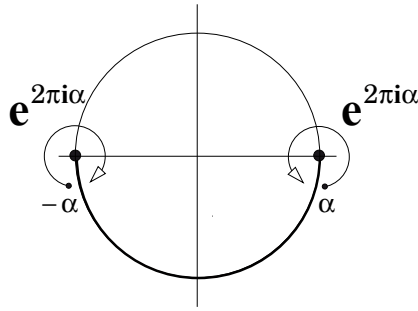


Exact formula

$$\beta(\alpha) = \frac{1}{2} \Psi\left(\frac{\alpha}{2}\right) + \frac{1}{2} \Psi\left(\frac{1-\alpha}{2}\right) + \gamma + \ln 4 + \frac{\pi}{2 \sin \pi \alpha}$$

$$\Psi(z) = \Gamma'(z)/\Gamma(z)$$

Riemann-Hilbert method



$$\Psi^{(-)}(kr, \phi) = \sum_{m=-\infty}^{\infty} c_m J_{|m|}(kr) e^{im\phi}$$

$$\Psi_l^{(+)}(kr, \phi) = J_{|l|}(kr) e^{il\phi} + \sum_{n=-\infty}^{\infty} t_n(l) H_{|n|}^{(1)}(kr) e^{in\phi}$$

When $kR \rightarrow 0$, $(\Delta + k^2)\Psi = 0 \rightarrow \Delta\Psi = 0$ and

$$\Psi(r, \phi) \xrightarrow{kR \rightarrow 0} \tilde{\Phi}(z) + \tilde{\Phi}(-\bar{z})$$

Inside the circle the function $\tilde{\Phi}(z)$ has to be regular

$$\tilde{\Phi}^{(\text{inside})}(z) = \sum_{m=0}^{\infty} a_m \left(\frac{z}{R}\right)^m.$$

Wave function outside the circle

$$\Psi_l^{(+)}(kr, \phi) = \tilde{\Phi}^{(+)}(z) + \tilde{\Phi}^{(+)}(-\bar{z}) + \rho y_0(0) \ln \frac{r}{R}$$

$$\tilde{\Phi}^{(+)}(z) = \frac{1}{2}(1 + y_0(0)) + \sum_{n=1}^{\infty} y_{-n}(0) \left(\frac{R}{z}\right)^n$$

The logarithmic term appears because of the presence of $H_0^{(1)}(kr)$

$$\ln \left(\frac{r}{R}\right) = \frac{1}{2} \ln \left(-i \frac{z}{R}\right) + \frac{1}{2} \ln \left(i \frac{\bar{z}}{R}\right)$$

Outside the circle

$$\Psi^{(+)} = \tilde{\Phi}^{(\text{outside})}(z) + \tilde{\Phi}^{(\text{outside})}(-\bar{z})$$

with

$$\tilde{\Phi}^{(\text{outside})}(z) = \tilde{\Phi}^{(+)}(z) + \frac{1}{2} \rho y_0(0) \ln \left(-i \frac{z}{R}\right)$$

$\tilde{\Phi}^{(+)}(z)$ is regular outside the circle but $\ln(-iz)$ is not

Aharonov-Bohm boundary conditions

Inside and outside functions of variable z (and of $-\bar{z}$) are related

$$\tilde{\Phi}^{(\text{inside})}(e^{i\phi}) = f(\phi)\tilde{\Phi}^{(\text{outside})}(e^{i\phi})$$

It is equivalent to

$$\begin{aligned}\tilde{\Phi}^{(-)}(e^{i\phi}) &= f(\phi)\tilde{\Phi}^{(+)}(e^{i\phi}) + \frac{\rho}{2}y_0(0)f(\phi)\ln(-ie^{i\phi}) \\ &\equiv f(\phi)\tilde{\Phi}^{(+)}(e^{i\phi}) + \rho y_0(0)f(\phi)g(\phi)\end{aligned}$$

where

$$\frac{1}{2}\ln(-ie^{i\phi}) \equiv g(\phi) = \frac{i}{2} \begin{cases} \phi - \frac{\pi}{2} & 0 < \phi < \pi \\ \phi - \frac{3\pi}{2} & \pi < \phi < 2\pi \end{cases}$$

Here $\tilde{\Phi}^{(-)}(e^{i\phi})$ and $\tilde{\Phi}^{(+)}(e^{i\phi})$ are boundary values of functions holomorphic inside and outside the circle

Non-homogeneous Riemann–Hilbert problem

Find two functions, $\tilde{\Phi}^{(-)}(z)$ and $\tilde{\Phi}^{(+)}(z)$, holomorphic, respectively, inside and outside a contour such that on the contour (parametrized by t) they obey

$$\tilde{\Phi}^{(-)}(t) = G(t)\tilde{\Phi}^{(+)}(t) + g(t)$$

The solution consists of two steps.

Step one: find solution of homogeneous problem

$T^{(-)}(t) = G(t)T^{(+)}(t)$. It reduced to

$$\ln T^{(+)}(t) - \ln T^{(-)}(t) = -\ln G(t)$$

whose solution is (\approx) given by the Cauchy integral

$$\ln T(z) = -\frac{1}{2\pi i} \oint \frac{\ln G(t)}{t-z} dt + \text{polynom}(z)$$

It means that

$$G(t) = \frac{T^{(-)}(t)}{T^{(+)}(t)}$$

and the non-homogeneous equation takes the form

$$\tilde{\Phi}^{(-)}(t) = \frac{T^{(-)}(t)}{T^{(+)}(t)} \tilde{\Phi}^{(+)}(t) + g(t)$$

Divide by $T^{(-)}(t)$

$$\frac{\tilde{\Phi}^{(-)}(t)}{T^{(-)}(t)} = \frac{\tilde{\Phi}^{(+)}(t)}{T^{(+)}(t)} + \frac{g(t)}{T^{(-)}(t)}$$

Its solution is again given by the Cauchy integral

$$\frac{\tilde{\Phi}(z)}{T(z)} = -\frac{1}{2\pi i} \oint \frac{g(t)}{(t-z)T^{(-)}(t)} dt + \text{polynom}(z)$$

etc.

Exact results

The S - wave function for the scattering on two vortices

$$\Psi_0^{(+)}(r, \phi) = J_0(kr) + J_0(kR) \sum_{n=-\infty}^{\infty} \frac{y_n(0)}{H_n^{(1)}(kR)} H_n^{(1)}(kr) e^{in\phi}$$

When $kR \rightarrow 0$

$$y_0(0) = -\frac{1}{1 + \beta(\alpha)\rho}$$

where

$$\rho = \frac{2i}{\pi + 2i(\ln(kR/2) + \gamma)} .$$

and

$$\beta(\alpha) = \frac{1}{2}\Psi\left(\frac{\alpha}{2}\right) + \frac{1}{2}\Psi\left(\frac{1-\alpha}{2}\right) + \gamma + \ln 4 + \frac{\pi}{2 \sin \pi\alpha}$$

The generating function of $y_n(0) = y_{-n}(0)$ in the limit $kR \rightarrow 0$ is

$$\sum_{n=0}^{\infty} y_n(0) z^n = \frac{1 + y_0}{2} \Phi(z) - \frac{1}{2}$$

where

$$\Phi(z) = \left(\frac{1+z}{1-z} \right)^\alpha \frac{J(1) - J(z)}{J(1) - J(0)}$$

and

$$J(z) = \frac{i}{4\pi} \int_0^\pi \left[e^{-i\pi\alpha/2} \frac{\tan^\alpha(\phi/2)}{1 - ze^{-i\phi}} + e^{i\pi\alpha/2} \frac{\cot^\alpha(\phi/2)}{1 + ze^{-i\phi}} \right] \left(\phi - \frac{\pi}{2} \right) d\phi$$

Contradict to Y. Nambu (Nucl. Phys. B **579** 590 (2000))

Summary

- Electromagnetic potentials can be observed in quantum mechanics even when the fields are zero everywhere except one point
- Scattering on a flux line exists when the flux is not an integer
- Collimated beam deflects on a flux line
- Flux line inside the billiard may change statistical properties
- Scattering on two near-by flux lines can be calculated analytically